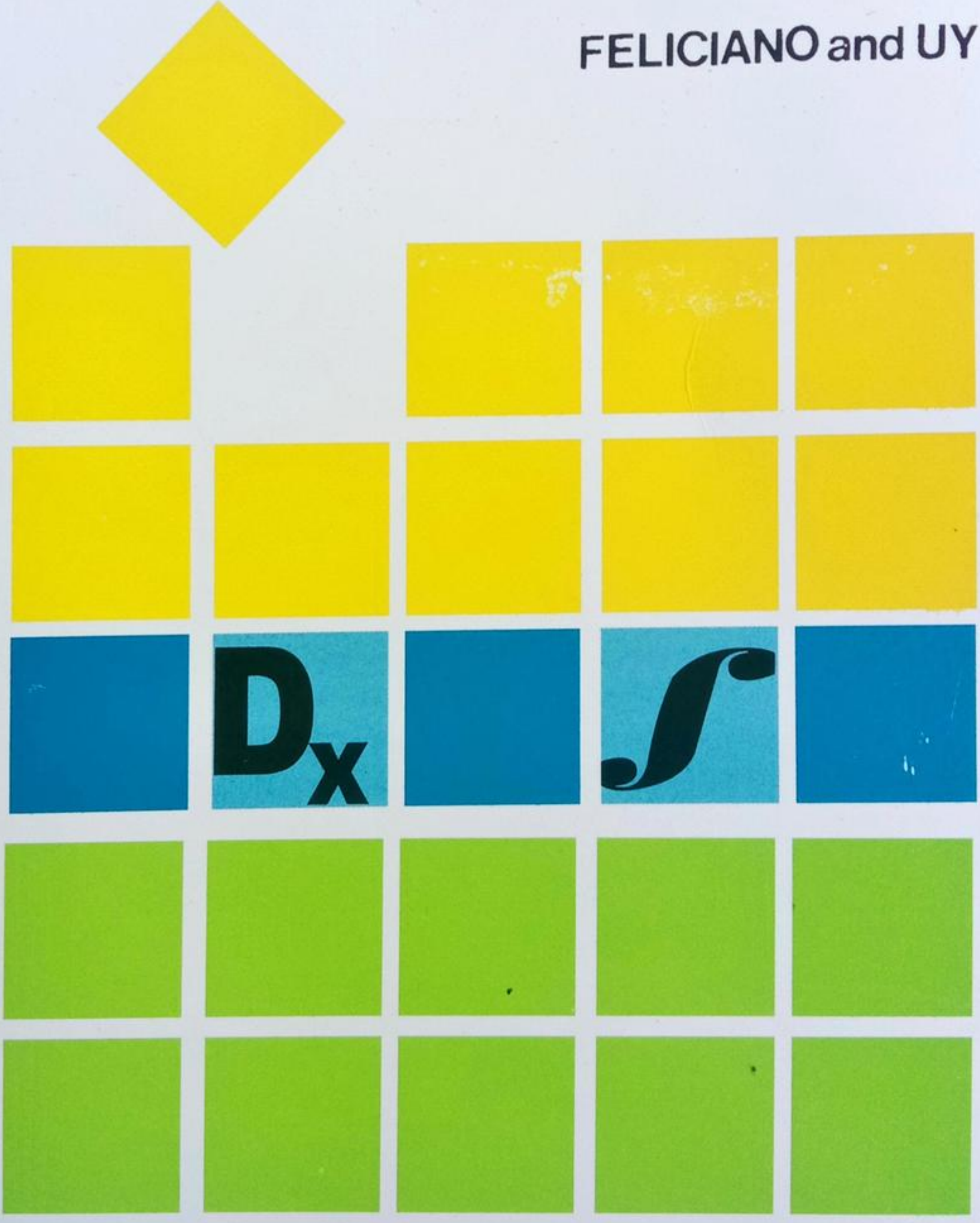
Differential & Integral Calculus

FELICIANO and UY



DIFFERENTIAL AND INTEGRAL CALCULUS

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Published By

MERRIAM & WEBSTER BOOKSTORE, INC.
Manila, Philippines

ISBN 971-30-0588-0

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preface

Based on the arrangement of the material, this book can be used as a text for a one-semester course in Differential Calculus (Chapter 1 to Chapter 8) and another one-semester course in Integral Calculus (Chapter 9 to Chapter 14). However, this present edition does not contain all the topics that are usually found in other books on calculus at the university level. This is justified for one simple reason. We believe that the necessary time to discuss all of them is not available. But then the instructor should feel free to discuss any topic not included in this book if time permits for such topic to be studied profitably.

In writing this book, special attention is given to the problem of presenting the material interestingly and understandably. To offset some of the difficulties which many students usually find in their first acquaintance with the principles and techniques of calculus, we tried to keep our discussion as simple, clear and informal as possible. We believe that too little rigor does not help students attain mathematical maturity and too much of it overwhelms them. To escape this regrettable possibility, rigor is judiciously used here. In fact, it should be noted that the formal proofs of some theorems which are essentially difficult were deliberately omitted. We have also included a large number of examples to illustrate the uses or applications of certain concepts, definitions or theorems but unless we feel that they are needed, we do not put in so many details in our illustrations. The problems in the exercises vary in difficulty. While there are problems which are obviously easy, there are also those which are relatively difficult.

Perhaps, to a certain degree, the ordering of the topics and the method of presentation differ from those of other books bearing the same name. Such changes arise largely from our experience. But our idea of what is best and of what goals are attainable now may undergo changes with the years. This is because the teaching of calculus is not static. For this fact, we encourage the instructor to formulate his own approach or method which he thinks is more teachable and understandable.

Our colleagues have been very cooperative and generous in sharing with us their observations and opinions regarding the weaknesses and strengths of our students in their study of calculus. Their encouragement and confidence in us have given us the spark and the patience to write this book. We are deeply grateful to them.

With this book in its final form, we hope that we have, in our small way, filled up the need for a clearer, more interesting and profitable text on calculus. However, despite considerable efforts by us, the publisher and the printer, some errors may still be found in this first edition. Any attention given us in this regard will be highly appreciated.

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1

Limits

Calculus is a branch of mathematics which started to develop in the 17th century. *Sir Isaac Newton* (English, 1642-1727) and *Wilhelm Leibniz* (German, 1646-1716) created calculus independently of each other and amazingly at about the same time. The invention of Calculus by these two remarkable men has provided the scientists a mathematics that could handle many of the difficult problems about motion and things that change. Today, calculus has important applications in almost every field of study that uses mathematics.

The word "calculus" is derived from the latin word for *stone* or *pebble*. In ancient times, pebbles were used for counting. Thus calculus roughly means a "method of calculation." The technique of calculation developed by Newton and Leibniz was undoubtedly remarkable so much so that it was called the CALCULUS.

In the study of calculus, the first important concept or idea which must be introduced is the concept of *limit*. Actually, as we shall eventually notice, the whole structure of calculus is based upon the limit concept. The purpose of this chapter is to impart to the student a thorough knowledge and understanding of this basic concept.

1.1 Functional Notation

The function concept is needed when we discuss the principles of calculus in general terms. Recall that a function is a certain law of correspondence. It is generally associated with a formula. For instance, consider the formula for the area of a circle

$$A = \pi r^2$$

For each value assigned to r, there corresponds a value of A. We say that A is a function of r and in symbol, we write

$$A = f(r).$$

In general, if there is a relation between two variables x and y such that for each value of x, there corresponds a value of y, then y is said to be a function of x. Symbolically, this is written in the form*

$$y = f(x)$$

The function concept may be extended to relations between more than two variables. Consider the equation

$$z = f(x,y).$$

This implies that z is determined when x and y are given and it is customary to say that z is a function of x and y. For instance, the volume of a right circular cylinder is a function of the altitude h and radius r of the base, that is,

$$V = f(r,h) = \pi r^2 h.$$

It is important that we be familiar with the *functional notation* In mathematics and the physical sciences functional notation plays a convenient and important part. In the example below, we shall illustrate how to set up a formula showing the functional relation between the variables.

EXAMPLE: The area of a rectangle is 6 sq. in. Express the perimeter P of the rectangle as a function of the length x of one side.

SOLUTION: Since the area is 6 sq. in., then the length of the other side is $\frac{6}{x}$ and the perimeter is

$$P = 2\left(x + \frac{6}{x}\right)$$

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*The notation $y = f(x)$ is due to the Swiss mathematician Leonard Euler (1707 1783).

EXERCISE 1.1

1. If $f(x) = x^2 - 4x$, find (a) $f(-5)$ (b) $f(y^2 + 1)$ (c) $f(x + \Delta x)$ (d) $f(x + 1) - f(x - 1)$.

2. If $y = \frac{x^2 + 3}{x}$, find x as a function of y.

3. If $y = \tan(x + \pi)$, find x as a function of y.

4. Express the distance D traveled in t hr by a car whose speed is 60 km/hr.

5. Express the area A of an equilateral triangle as a function of its side x.

6. The stiffness of a beam of rectangular cross section is proportional to the breadth and the cube of the depth. If the breadth is 20 cm, express the stiffness as a function of the depth.

7. A right circular cylinder, radius of base x, height y, is inscribed in a right circular cone, radius of base r and height h Express y as function of x (r and h are constants).

8. If $f(x) = x^2 + 1$, find $\frac{f(x+h) - f(x)}{h}$, $h \neq 0$.

9. If $f(x) = 3x^2 - 4x + 1$, find $\frac{f(h + 3) - f(3)}{h}$, $h \neq 0$.

10. If $f(x) = \frac{4}{x + 3}$ and $g(x) = x^2 - 3$, find $f\left[g(x)\right]$ and $g\left[f(x)\right]$

1.2 Limit of a Function

Familiarity with the limit concept is absolutely essential for a deeper understanding of the calculus. In this section, we shall begin our discussion of the limit of a function but we emphasize that our treatment here will appeal more to our intuition than to rigor. And since our approach is a non-rigorous one, we therefore, expect you to grasp this idea with ease.

Consider the function defined by the equation

$$f(x) = 3x + 1$$

and assign some values to x near, but not equal, to a specific number, say 2. For each value of x in the neighborhood of 2, we compute the corresponding value of y. To get an idea of what is happening, we construct a table of values as shown below:

x	1.500	1.890	1.999	2.009	2.050	2.160	2.300	
$f(x)$	5.500	6.670	6.997	7.027	7.150	7.480	7.900	

The table shows that when x is near 2, whether a little less or a little greater than 2, $f(x) = 3x+1$ is nearer 7. In other words, "*$3x+1$ approaches the number 7 as a limit when x approaches 2*" The abbreviated symbolic form for this statement is

$$3x+1 \to 7 \quad \text{as} \quad x \to 2.$$

We may also say that "*the limit of $3x+1$ as x approaches 2 is 7.*" In symbol, we write this as

$$\lim_{x \to 2} (3x+1) = 7.$$

From our intuitive discussion above, we may formulate the following definition of the limit of a function.

DEFINITION 1.1 Let $f(x)$ be any function and let a and L be numbers. If we can make $f(x)$ as close to L as we please by choosing x sufficiently close to a then we say that the limit of $f(x)$ as x approaches a is L or symbolically.

$$\lim_{x \to a} f(x) = L.$$

The definition above is, of course, quite imprecise since it is based purely on an intuitive point of view*. Ho[illegible]while intuition is not always convincing to everyone, we think that the given definition is adequate for our purpose aside from the fact that our approach here saves a good deal of development. In fact, we should mention that, historically, the limit concept developed or evolved in just the way we have presented it above.

1.3 Theorems on Limits

This section deals with several theorems by means of which we shall be able to evaluate the limits of functions rapidly and efficiently. To evaluate or to find

$$\lim_{x \to a} f(x)$$

means that we are to find the number L that f(x) is near, whenever x is near a but not aqual to a. Of course, when $x = a$, the value of the function is $f(a)$. It may be that $f(a)$ is also the limit, i.e., $L = f(a)$. Thus to evaluate

$$\lim_{x \to 1} (4 - x^2)$$

means to find a number which $4-x^2$ is near whenever x is near the number 1. By Definition 1.1, we know that

$$\lim_{x \to 1} (4 - x^2) = 3$$

since by choosing x sufficiently close to 1, $4-x^2$ can be made to come as close to 3 as we please.

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[*] A precise definition of limit can be found in more advanced calculus text.

To obtain the limits of more complicated functions, we shall use the following theorems which we shall state symbolically without proof.*

L1. $\lim_{x \to a} c = c$, c = any constant

L2. $\lim_{x \to a} x = a$, a = any real number

L3. $\lim_{x \to a} c\,f(x) = c \lim_{x \to a} f(x)$

L4. $\lim_{x \to a} \left[f(x) + g(x)\right] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

L5. $\lim_{x \to a} \left[f(x) \cdot g(x)\right] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

L6. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

L7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, n = any positive integer and $f(x) \geq 0$ if n is even.

L8. $\lim_{x \to a} \left[f(x)\right]^n = \left[\lim_{x \to a} f(x)\right]^n$

In stating the above theorems, we assume that $f(x)$ and $g(x)$ are defined for all values of x in some interval containing a, except possibly at a itself. These theorems may be stated briefly in words. For instance, L4 is sometimes stated as "*the limit of a sum is the sum of the limits.*" To illustrate the use of these theorems, we have the following examples.

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*The proofs of these theorems are made by use of the formal definition of a limit (delta-epsilon definition) which is not within the scope of this book.

EXAMPLE 1:

$$\lim_{x \to 2} (x^2 + 3x + 4) = \lim_{x \to 2} x^2 + \lim_{x \to 2} 3x + \lim_{x \to 2} 4 \qquad \text{by L4}$$

$$= \left[\lim_{x \to 2} x\right]^2 + 3\lim_{x \to 2} x + 4 \qquad \text{by L8, L3, L1}$$

$$[2]^2 + 3(2) + 4 \qquad \text{by L2}$$

$$= 14$$

EXAMPLE 2:

$$\lim_{x \to 2} (x + 4)\sqrt{2x + 5} = \lim_{x \to 2} (x + 4) \lim_{x \to 2} \sqrt{2x + 5} \qquad \text{by L5}$$

$$= (\lim_{x \to 2} x + \lim_{x \to 2} 4)\sqrt{\lim_{x \to 2} (2x + 5)} \qquad \text{by L4, L7}$$

$$= (\lim_{x \to 2} x + \lim_{x \to 2} 4)\sqrt{\lim_{x \to 2} 2x + \lim_{x \to 2} 5} \qquad \text{by L4}$$

$$= (\lim_{x \to 2} x + \lim_{x \to 2} 4)\sqrt{2\lim_{x \to 2} x + \lim_{x \to 2} 5} \qquad \text{by L3}$$

$$= (2 + 4)\sqrt{2(2) + 5} \qquad \text{by L2, L1}$$

$$= 18$$

EXAMPLE 3:

$$\lim_{x \to 3} (3x + 4)^2 = \left[\lim_{x \to 3} (3x + 4)\right]^2 \qquad \text{by L8}$$

$$= \left[\lim_{x \to 3} 3x + \lim_{x \to 3} 4\right]^2 \qquad \text{by L4}$$

$$= \left[3\lim_{x \to 3} x + \lim_{x \to 3} 4\right]^2 \qquad \text{by L3}$$

$$= \left[3(3) + 4\right]^2 \qquad \text{by L2, L1}$$

$$= 169$$

Note that the limits of the functions in the above examples can be obtained by *straight substitution*. For instance, in example 2, we see that straight substitution of $x = 2$ gives the desired limit. Thus in practice, the solution may simply be written as follows:

$$\begin{aligned} \lim_{x \to 2} (x+4)\sqrt{2x+5} &= (2+4)\sqrt{2(2)+5} \\ &= (6)\sqrt{9} \\ &= 18 \end{aligned}$$

EXERCISE 1.2

Evaluate each of the following:

1. $\lim_{x \to 2} (x^2 - 4x + 3)$
2. $\lim_{x \to 3} \frac{4x + 2}{x + 4}$
3. $\lim_{x \to \frac{\pi}{4}} (\tan x + \sin x)$
4. $\lim_{x \to \frac{\pi}{3}} \frac{\sin 2x}{\sin x}$
5. $\lim_{x \to 8} (2x + \sqrt[3]{x} - 4)$
6. $\lim_{x \to 2} (4x - 3)(x^2 + 5)$
7. $\lim_{x \to 3} \frac{\sqrt{3x}}{x\sqrt{x + 1}}$
8. $\lim_{x \to 0} \frac{3x + 2}{x^2 - 2x + 4}$

1.4 Indeterminate Forms

Consider the function defined by

$$f(x) = \frac{N(x)}{D(x)} \qquad D(x) \neq 0$$

Suppose at $x = a$, $N(a) = D(a) = 0$

$$f(a) = \frac{N(a)}{D(a)} = \frac{0}{0}$$

which is undefined. We say that at $x = a$, the function $f(x)$ assumes the indeterminate form $\frac{0}{0}$. The other indeterminate form* that we shall encounter here is $\frac{\infty}{\infty}$. Obtaining any of these forms by straight substitution does not necessarily mean that $f(x)$ has no limit. We shall see in the examples below that even if $f(x)$ assumes the indeterminate form $\frac{0}{0}$ at $x = a$, the limit of $f(x)$ may be definite, i.e., the limit exists. This limit is usually found by changing the expression defined by $f(x)$ into a form to which the theorems on limits can be used. Consider the following examples.

EXAMPLE 1. Evaluate $\lim\limits_{x \to 2} \frac{x^2 - 4}{x - 2}$

Solution: This can not be evaluated by straight substitution since when $x = 2$, we have

$$\frac{x^2 - 4}{x - 2} = \frac{4 - 4}{2 - 2} = \frac{0}{0}$$

which is meaningless. That is, at $x = 2$, the function assumes the indeterminate form $\frac{0}{0}$ However, if $x \neq 2$, then

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2$$

Therefore, to evaluate the limit of the given function, we proceed as follows:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2}$$

$$= \lim_{x \to 2} (x + 2)$$

$$= 2 + 2$$

$$= 4$$

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*Other indeterminate forms will be discussed in Chapter 5.

The example above illustrates the fact that $f(x)$ may have a limit at a number a even though the value $f(a)$ of the function is undefined. Moreover, it shows that the limit and value of the function are two different concepts.

EXAMPLE 2. Evaluate $\lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$ if $f(x) = x^2 - 3x$

Solution: A straight substitution of $x = 2$ leads to the indeterminate form $\frac{0}{0}$. Since $f(x) = x^2 - 3x$, then $f(2) = 4 - 6 = -2$. Hence.

$$\begin{aligned}
\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \to 2} \frac{(x^2 - 3x) - (-2)}{x - 2} \\
&= \lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} \\
&= \lim_{x \to 2} \frac{(x - 1)(x - 2)}{x - 2} \\
&= \lim_{x \to 2} (x - 1) \\
&= 1
\end{aligned}$$

EXERCISE 1.3

Evaluate each of the following:

1. $\lim\limits_{x \to 4} \dfrac{x^3 - 64}{x^2 - 16}$
2. $\lim\limits_{x \to 2} \dfrac{x^2 + 2x - 8}{3x - 6}$
3. $\lim\limits_{x \to 3} \dfrac{x^3 - 13x + 12}{x^3 - 14x + 15}$
4. $\lim\limits_{x \to 2} \dfrac{x^3 - x^2 - x - 2}{2x^3 - 5x^2 + 5x - 6}$

5. $\lim\limits_{x \to 0} \dfrac{(x+3)^2 - 9}{2x}$

6. $\lim\limits_{x \to 0} \dfrac{\sqrt{x+16} - 4}{x}$

7. $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{x+3} - 2}$

8. $\lim\limits_{x \to 8} \dfrac{\sqrt[3]{x} - 2}{x - 8}$

9. $\lim\limits_{x \to 4} \dfrac{\frac{1}{x} - \frac{1}{4}}{x - 4}$

10. $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^2 - 4}$

11. $\lim\limits_{x \to 3} \dfrac{x - 3}{\sqrt{x-2} - \sqrt{4-x}}$

12. $\lim\limits_{x \to 0} \dfrac{1}{x} \left(\dfrac{1}{3} - \dfrac{1}{\sqrt{x+9}} \right)$

13. $\lim\limits_{x \to 3} \dfrac{\sqrt{x^2 - 9}}{x - 3}$

14. $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\tan 2x}{\sec 2x}$

15. $\lim\limits_{x \to 0} \dfrac{\sin^3 x}{\sin x - \tan x}$

16. $\lim\limits_{x \to 0} \dfrac{1 - \cos^2 x}{1 + \cos x}$

17. $\lim\limits_{x \to 0} \dfrac{\sin x \sin 2x}{1 - \cos x}$

18. $\lim\limits_{x \to \pi} \dfrac{\sin^2 x}{1 + \cos x}$

If $f(x) = \sqrt{x}$, find

19. $\lim\limits_{x \to 4} \dfrac{f(x) - f(4)}{x - 4}$

20. $\lim\limits_{x \to 0} \dfrac{f(9 + x) - f(9)}{x}$

If $f(x) = x^2 - 2x + 3$, find

21. $\lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$

22. $\lim\limits_{x \to 0} \dfrac{f(x + 2) - f(2)}{x}$

1.5 Infinity

Let $f(x)$ be a function. If we can make $f(x)$ as large as we please by making x close enough, but not equal, to a real number a, then we describe this situation by writing.

$$\lim_{x \to a} f(x) = \infty$$

where the symbol ∞ is read "infinity".

In particular, consider the function $f(x) = \dfrac{1}{x}$. The table below shows that as x takes on values successively approaching the number 0, the value of $\dfrac{1}{x}$ grows larger and larger. We say that $\dfrac{1}{x}$ becomes infinite as x approaches 0 and indicate this by writing

$$\frac{1}{x} \to \infty \quad \text{as} \quad x \to 0$$

In more compact form, we write

$$\lim_{x \to 0} \frac{1}{x} = \infty$$

x	0.1000	0.0100	0.0010	0.0001	→ 0
$f(x) = \frac{1}{x}$	10	100	1,000	10,000	→ ∞

Bear in mind that ∞ is not a number which results from division by zero. Recall that in the real number system, division by zero is not permissible. In fact, it can be argued that the statement

$$\lim_{x \to a} f(x) = \infty$$

is not an equation at all since ∞ does not represent a number. It is merely used as a symbol to imply that the value of f(x) increases numerically without bound as x approaches a.*

1.6 Limit at Infinity

A function f(x) may have a finite limit even when the independent variable x becomes infinite. This statement "x becomes infinite" is customarily expressed in symbolism by "$x \to \infty$".

Consider again the function $f(x) = \frac{1}{x}$. It can be shown (intuitively or formally) that $\frac{1}{x}$ approaches a finite limit (the number zero) as x increases without bound. That is,

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*The symbol ∞ is used for infinity if no particular reference to sign is made. The symbols $+\infty$ (read "plus infinity") and $-\infty$ (read "minus infinity") are used in some books in connection with statements about limits. The symbol $+\infty$ is used to indicate that f (x) becomes positively infinite (increases without bound) while $-\infty$ is used to mean that f (x) becomes negatively infinite (decreases without bound).

$$\frac{1}{x} \to 0 \text{ as } x \to \infty$$

We shall consider this fact as an additional theorem on limits and in symbol, we write

$$\text{L9} \qquad \lim_{x \to \infty} \frac{1}{x} = 0$$

The use of L9 is illustrated in the following examples.

EXAMPLE 1. $\displaystyle\lim_{x \to \infty} \frac{1}{x^3} = \lim_{x \to \infty} \left(\frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x}\right)$

$$= \lim_{x \to \infty} \frac{1}{x} \cdot \lim_{x \to \infty} \frac{1}{x} \cdot \lim_{x \to \infty} \frac{1}{x} \qquad \text{by L5}$$

$$= 0$$

EXAMPLE 2. $\displaystyle\lim_{x \to \infty} \frac{4}{x^2} = 4 \lim_{x \to \infty} \frac{1}{x^2}$ why?

$$= 4 \lim_{x \to \infty} \left(\frac{1}{x} \cdot \frac{1}{x}\right)$$

$$= 4 \lim_{x \to \infty} \frac{1}{x} \cdot \lim_{x \to \infty} \frac{1}{x} \qquad \text{by L5}$$

$$= 0 \qquad \text{by L9}$$

EXAMPLE 3. $\displaystyle\lim_{x \to \infty} \frac{1}{x^{\frac{1}{4}}} = \lim_{x \to o} \left(\frac{1}{x}\right)^{\frac{1}{4}}$ why?

$$= \left[\lim_{x \to \infty} \frac{1}{x}\right]^{\frac{1}{4}} \qquad \text{by L8}$$

$$= 0 \qquad \text{by L9}$$

From the examples above, we intuitively feel that if n is any positive number, then

$$\lim_{x \to \infty} \frac{1}{x^n} = 0$$

This is given as a theorem in some books. Note that when n= 1, we have L9.

A function $f(x) = \frac{N(x)}{D(x)}$ may assume the indeterminate form $\frac{\infty}{\infty}$ when x is replaced by ∞. However, the limit of f(x) as x becomes infinite may be definite. To find this limit we first divide N(x) and D(x) by the highest power of x. Then we evaluate the limit by use of L9.

EXAMPLE: Evaluate $\lim_{x \to \infty} \frac{4x^3 + 3x^2 - 6}{2x^3 + 5x + 3}$

Solution: The function assumes the indeterminate form $\frac{\infty}{\infty}$ when x is replaced by ∞. Dividing the numerator and denominator by x^3, we get

$$\lim_{x \to \infty} \frac{4x^3 + 3x^2 - 6}{2x^3 + 5x + 3} = \lim_{x \to \infty} \frac{4 + \frac{3}{x} - \frac{6}{x^3}}{2 + \frac{5}{x^2} + \frac{3}{x^3}}$$

$$= \frac{5 + 0 - 0}{2 + 0 + 0}$$

$$= 2$$

EXERCISE 1.4

Evaluate each of the following.

1. $\lim_{x \to \infty} \frac{6x^3 + 4x^2 + 5}{8x^3 + 7x - 3}$

5. $\lim_{x \to \infty} \frac{8x - 5}{\sqrt{4x^2 + 3}}$

2. $\lim_{x \to \infty} \dfrac{3x^2 + x + 2}{x^3 + 8x + 1}$

3. $\lim_{x \to \infty} \dfrac{4x + 5}{x^2 + 1}$

4. $\lim_{x \to \infty} \dfrac{x^3 + x + 2}{x^2 - 1}$

6. $\lim_{x \to \infty} \dfrac{x^3}{(2x - 1)^2}$

7. $\lim_{x \to \infty} \dfrac{(x + 2)^3 - (x - 2)^3}{x^2}$

8. $\lim_{x \to \infty} \dfrac{\sqrt{9x^2 + 4}}{6x + 1}$

1.7 Continuity

In Section 1.4, we emphasized that the *limit* and *value* of a function are two different concepts. In fact, in Section 1.2, when we discussed the meaning of $\lim_{x \to a} f(x) = L$, we deliberately ignored the actual value of f(x) at $x = a$. However, in Section 1.3, we made mention of the fact that the limit of a function f(x) as $x \to a$ may turn out to be just the value of f(x) at $x = a$. That is, $\lim_{x \to a} f(x) = f(a)$. Now when this happens, we have an event of some mathematical significance. The function f(x) is said to be *continuous* at $x = a$. This leads to the following definition.

DEFINITION 1.2 A function f(x) is continuous at $x = a$ if if $\lim_{x \to a} f(x) = f(a)$.

Note that the condition $\lim_{x \to a} f(x) = f(a)$ in the definition[*] above actually implies three conditions, namely

(1) $f(a)$ is defined.

(2) $\lim_{x \to a} f(x) = L$ exists, and

(3) $L = f(a)$

If any of these conditions is not satisfied, then f(x) is said to be *discontinuous* at $x = a$.

A function f(x) is said to be *continuous in an interval if it is continuous for every value of x in the interval.* The graph of

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[*] This definition was formulated by the French mathematician Augustin Louis Cauchy (1789-1857).

this function is "unbroken" over that interval. That is, the graph of f(x) can be drawn without lifting the pencil from the paper (see Fig. 1.1).

EXAMPLE 1. The function $f(x) = x^2$ is continuous at $x = 2$ because $\lim_{x \to 2} x^2 = f(2) = 4$. In fact, it is continuous for all finite values of x. The graph of the function is shown in Fig. 1.1.

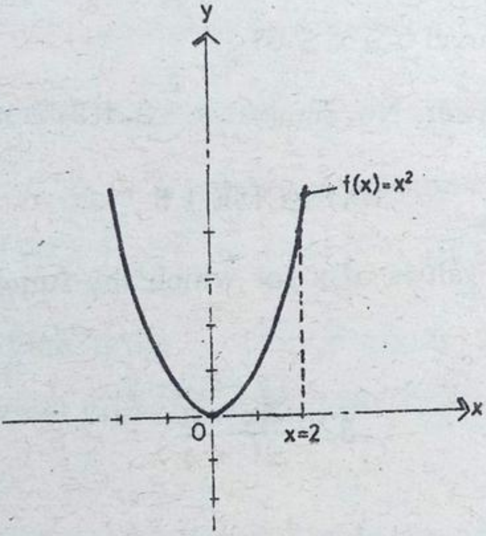

FIG. 1.1

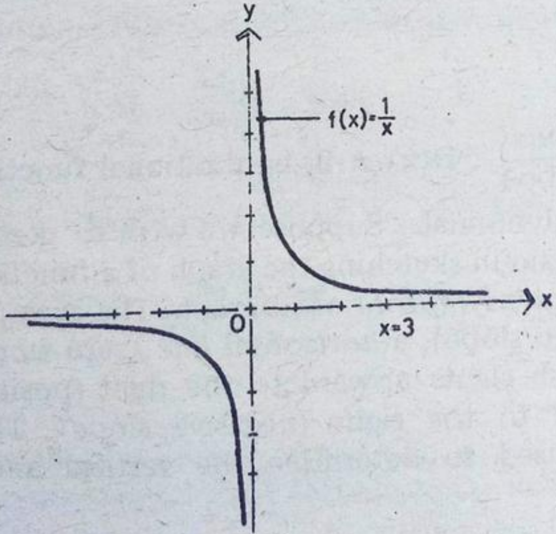

FIG. 1.2

EXAMPLE 2. The function $f(x) = \frac{1}{x}$ is continuous at $x = 3$ because $\lim_{x \to 3} \frac{1}{x} = f(3) = \frac{1}{3}$. It is, however, discontinuous at $x = 0$ since $\lim_{x \to 0} \frac{1}{x} = \infty$. The graph of the function (see Fig. 1.2) contains a "break" at $x = 0$.

EXAMPLE 3. Is the function $f(x) = \frac{4x}{x^2 - 4}$ continuous over the interval $0 \leq x \leq 5$?

Answer: No, since at $x = 2$, $f(2)$ is undefined.

EXERCISE 1.5

Find the value or values of x for which the function is discontinuous.

1. $\frac{3x}{x-5}$
2. $\frac{3x + 2}{x^2 - 8x + 15}$
3. $\frac{5x + 1}{x^2 + 4}$
4. $\frac{6x}{x^2 - 9}$
5. $\frac{1}{2^x - 8}$
6. $\frac{x + 3}{x^3 - 3x^2 + 2x}$

1.8 Asymptotes

Let $f(x) = \frac{N(x)}{D(x)}$, $D(x) \neq 0$, be a rational function, i.e. $N(x)$ and $D(x)$ are polynomials. Suppose we wish to sketch the graph of $f(x)$. A useful aid in sketching the graph of a function is to find, if there is any, the *asymptote* of its graph. The asymptote may be a *vertical line* (no slope), a *horizontal line* (zero slope) or a *non-vertical line* which slants upward to the right (positive slope) or slants downward to the right (negative slope). The following definitions are used to determine the vertical and horizontal asymptotes.

DEFINITION 1.3 The line $x = a$ is a vertical asymptote of the graph of $f(x)$ if $\lim_{x \to a} f(x) = \infty$.

DEFINITION 1.4 The line $y = b$ is a horizontal asymptote of the graph of $f(x)$ if $\lim_{x \to \infty} f(x) = b$.

EXAMPLE 1. Since $\lim_{x \to 3} \frac{2x}{x-3} = \infty$, then $x = 3$ is a vertical asymptote of the graph of the function defined by $f(x) = \frac{2x}{x-3}$.

EXAMPLE 2. $y = 2$ is a horizontal asymptote of the graph of $f(x) = \frac{4x^2}{2x^2-6}$ since $\lim_{x \to \infty} \frac{4x^2}{2x^2-6} = 2$.

EXAMPLE 3. $y = 0$ is a horizontal asymptote of the of $f(x) = \frac{3x}{x^2-1}$ since $\lim_{x \to \infty} \frac{3x}{x^2-1} = 0$.

EXAMPLE 4. There is no horizontal asymptote for the graph of $f(x) = \frac{4x^2}{2x-1}$ since $\lim_{x \to \infty} \frac{4x^2}{2x-1} = \infty$.

From Definitions 1.3 and 1.4 and the examples above, we can make certain generalizations which would facilitate further the process of finding the vertical and horizontal asymptotes* of the graph of the rational function defined by the equation

$$f(x) = \frac{N(x)}{D(x)}, \; D(x) \neq 0$$

Since $N(x)$ and $D(x)$ are polynomials, we may let

$$N(x) = a_0 x^m + a_1 x^{m-1} + \ldots + a_{m-1} + a_m$$

$$D(x) = b_0 x^n + b_1 x^{n-1} + \ldots + b_{n-1} + b_n$$

where m and n are positive integers and $a_0, a_1, \ldots a_m$ and $b_0, b_1,$ b_n are constants. We now formulate the following rules for

---

*Other properties of a curve such as its intercepts and symmetry are assumed familiar to the student.

finding the vertical and horizontal asymptotes of the rational function $f(x) = \frac{N(x)}{D(x)}$:

1. To find the vertical asymptote of the graph of $f(x)$, we set $D(x) = 0$ and solve for x. If $x = k$ where k is any real number, then the vertical asymptote is the line $x = k$.

2. To find the horizontal asymptote of the graph of $f(x)$, we have the following conditions to observe:

C1: If $m < n$, then the horizontal asymptote is $y = 0$.

C2: If $m = n$, then the horizontal asymptote is $y = \frac{a_o}{b_o}$.

C3: If $m > n$, then there is no horizontal asymptote.

EXAMPLE 5. Sketch the graph of $y = \frac{2x^2}{x^2 - 4}$.

Solution: Note that $N(x) = 2x^2$ and $D(x) = x^2 - 4$.

I. Intercepts: $x = 0$ and $y = 0$
II. Symmetry: Symmetric with respect to the y-axis.
III. Asymptotes:

(1) Setting $x^2 - 4 = 0$, we get $x = 2$ and $x = -2$ as vertical asymptotes.

(2) Since $m = n = 2$, condition C2 is satisfied and because $a_o = 2$ and $b_o = 1$, then the horizontal asymptote is the line $y = 2$.

IV. Additional Information:

(1) For all $x > 2$, we find that $y > 0$. The curve lies above the x-axis.

(2) For all $-2 < x < 2$, we find $y < 0$. The curve lies below the x-axis.

(3) For all $x < -2$, we find $y > 0$. The curve lies above the x-axis.
The graph of the function is shown in Fig. 1.3

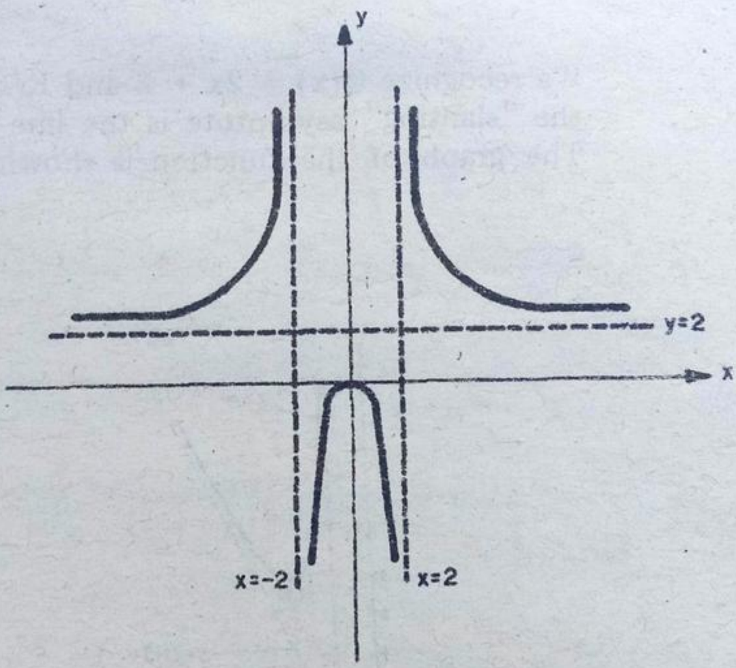

FIG. 1.3

Consider again condition C3 where $m > n$, i.e. where the degree of $N(x)$ is greater than the degree of $D(x)$. While there is no horizontal asympotote, there may be a "slanting" symptote. To find the equation of this asymptote, we proceed as follows:

By long division, write

$$y = \frac{N(x)}{D(x)}$$

in the form

$$y = Q(x) + \frac{R(x)}{D(x)}$$

where the degree of $R(x)$ is less than the degree of $D(x)$. The equation of the "slanting" asymptote is $y = Q(x)$.