

CHAPTER 1: LIMITS

EXERCISE 1.1

1. If $f(x) = x^2 - 4x$, find (a) $f(-5)$ (b) $f(y^2 + 1)$ (c) $f(x + \Delta x)$ (d) $f(x + 1) - f(x - 1)$

(a) $f(-1) = (-5)^2 - 4(-5)$

$= 25 + 20$

$f(-1) = 45$

(b) $f(y^2 + 1) = (y^2 + 1)^2 - 4(y^2 + 1)$

$= y^4 + 2y^2 + 1 - 4y^2 - 4$

$= y^4 - 2y^2 - 3$

$f(y^2 + 1) = (y^2 - 3)(y^2 + 1)$

(c) $f(x + \Delta x) = (x + \Delta x)^2 - 4(x + \Delta x)$

$= (x + \Delta x)[(x + \Delta x) - 4]$

$f(x + \Delta x) = (x + \Delta x)(x + \Delta x - 4)$

(d) $f(x + 1) - f(x - 1) = [(x + 1)^2 - 4(x + 1)] - [(x - 1)^2 - 4(x - 1)]$

$= [(x^2 + 2x + 1) - 4x - 4] - [(x^2 - 2x + 1) - 4x + 4]$

$= [x^2 - 2x - 3] - [x^2 - 6x + 5]$

$= x^2 - 2x - 3 - x^2 + 6x - 5$

$f(x + 1) - f(x - 1) = 4x - 8$ or $2(x - 4)$

2. If $y = \frac{x^2 + 3}{x}$, find x as a function of y

$y = \frac{x^2 + 3}{x}$

$x^2 + 3 = xy$

$x^2 - xy + 3 = 0$

Find the value of x by using the quadratic equation.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a = 1, b = -y, c = 3$

$x = \frac{-(-y) \pm \sqrt{(-y)^2 - 4(1)(3)}}{2(1)}$

$x = \frac{y \pm \sqrt{y^2 - 12}}{2}$

3 If $y = \tan(x + \pi)$, find x as a function of y .

$y = \tan(x + \pi)$

$x + \pi = \arctan(y)$

$x = \arctan(y) - \pi$

4. Express the distance D traveled in t hr by a car whose speed is 60 km/hr.

Distance = Rate \times Time

$D = 60t$

5. Express the area A of an equilateral triangle as a function of its side x .

$A = \frac{1}{2}bh$

$a^2 + b^2 = c^2$

$\left(\frac{1}{2}x\right)^2 + h^2 = (x^2)$

$\frac{1}{4}x^2 + h^2 = x^2$

$h^2 = x^2 - \frac{1}{4}x^2$

$h^2 = \frac{4x^2 - x^2}{4}$

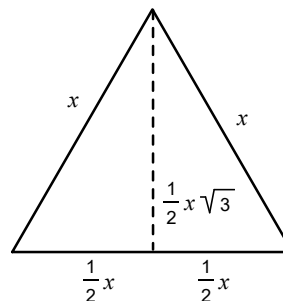
$h^2 = \frac{3x^2}{4}$

$h = \frac{\sqrt{3}}{2}x$

Let $b = x, h = \frac{\sqrt{3}}{2}x$

$A = \frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right)$

$A = \frac{\sqrt{3}}{4}x^2$

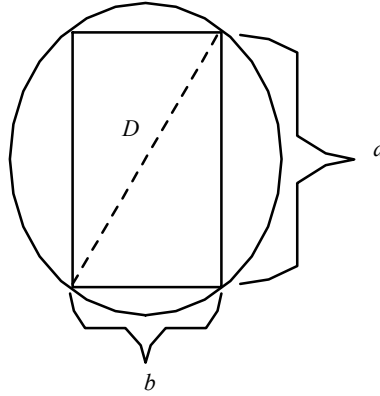


6. The stiffness of a beam of rectangular cross section is proportional to the breadth and the cube of the depth. If the breadth is 20 cm, express the stiffness as a function of the depth.

Let S = stiffness, b = breadth, d = depth

$$S = b d^3$$

$$S = 20 d^3$$



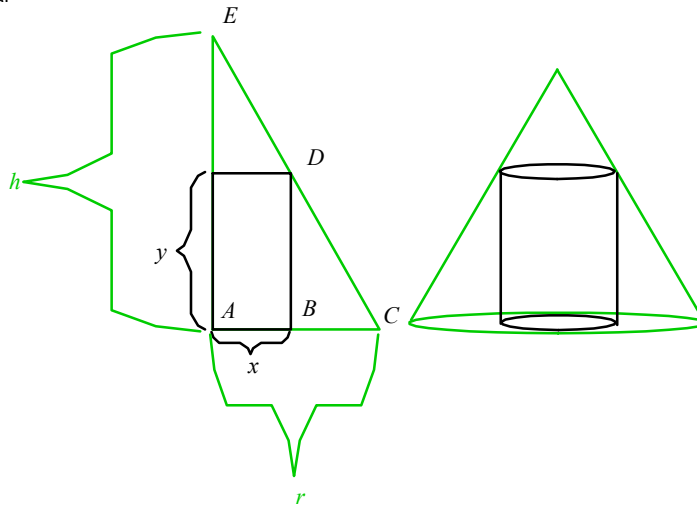
7. A right circular cylinder, radius of base x , height y , is inscribed in a right circular cone, radius of base r and height h . Express y as function of x (r and h are constant)

Using ratio and proportion of two similar triangle $\triangle BCD$ and $\triangle ACE$

$$\frac{y}{r-x} = \frac{h}{r}$$

$$ry = h(r-x)$$

$$y = \frac{h(r-x)}{r}$$



8. If $f(x) = x^2 + 1$, find $\frac{f(x+h) - f(x)}{h}$, $h \neq 0$

$$\frac{f(x+h) - f(x)}{h} = \frac{[(x+h)^2 + 1] - [x^2 + 1]}{h}$$

$$= \frac{[x^2 + 2xh + h^2 + 1] - x^2 - 1}{h}$$

$$= \frac{2xh + h^2}{h}$$

$$= \frac{h(2x+h)}{h}$$

$$\frac{f(x+h) - f(x)}{h} = 2x + h$$

9. If $f(3x^2 - 4x + 1)$, find $\frac{f(h+3) - f(3)}{h}$, $h \neq 0$

$$\frac{f(h+3) - f(3)}{h} = \frac{[3(h+3)^2 - 4(h+3) + 1] - [3(3)^2 - 4(3) + 1]}{h}$$

$$= \frac{[3(h^2 + 6h + 9) - 4h - 12 + 1] - [27 - 12 + 1]}{h}$$

$$= \left[\frac{3h^2 + 18h + 27 - 4h - 12 + 1 - 27 + 12 - 1}{h} \right]$$

$$= \frac{3h^2 + 14h}{h}$$

$$= \frac{h(3h+14)}{h}$$

$$\frac{f(h+3)-f(3)}{h} = 3h+14$$

10 If $f(x) = \frac{4}{x+3}$ and $g(x) = x^2 - 3$, find $f(g(x))$ and $g(f(x))$

$$f(g(x)) = \frac{4}{x^2 - 3 + 3}$$

$$f(g(x)) = \frac{4}{x^2}$$

$$g(f(x)) = \left(\frac{4}{x+3}\right)^2 - 3$$

$$g(f(x)) = \frac{16}{x^2 + 6x + 9} - 3$$

$$g(f(x)) = \frac{16 - (3x^2 + 18x + 27)}{x^2 + 6x + 9}$$

$$g(f(x)) = \frac{16 - 3x^2 - 18x - 27}{x^2 + 6x + 9}$$

$$g(f(x)) = \frac{-3x^2 - 18x - 11}{x^2 + 6x + 9}$$

$$g(f(x)) = \frac{-3x^2 - 18x - 11}{(x+3)^2}$$

EXERCISE 1.2

Evaluate each of the following.

1. $\lim_{x \rightarrow 2} (x^2 - 4x + 3)$

$$= (2)^2 - 4(2) + 3$$

$$= 4 - 8 + 3$$

$$\lim_{x \rightarrow 2} (x^2 - 4x + 3) = -1$$

2. $\lim_{x \rightarrow 3} \left(\frac{3x+2}{x+4} \right)$

$$= \frac{3(3)+2}{3+4}$$

$$= \frac{9+2}{7}$$

$$\lim_{x \rightarrow 3} \left(\frac{3x+2}{x+4} \right) = \frac{11}{7}$$

3. $\lim_{x \rightarrow \frac{\pi}{4}} (\tan x + \sin x)$

$$= \tan \frac{\pi}{4} + \sin \frac{\pi}{4}$$

$$= 1 + \frac{\sqrt{2}}{2}$$

$$\lim_{x \rightarrow \frac{\pi}{4}} (\tan x + \sin x) = \frac{2 + \sqrt{2}}{2}$$

4. $\lim_{x \rightarrow \frac{\pi}{3}} \left(\frac{\sin 2x}{\sin x} \right)$

$$= \frac{\sin 2\left(\frac{\pi}{3}\right)}{\sin \frac{\pi}{3}}$$

$$= \frac{\sin \frac{2\pi}{3}}{\sin \frac{\pi}{3}}$$

$$= \frac{\sin \frac{2\pi}{3}}{\sin \frac{\pi}{3}}$$

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= \frac{\sqrt{3}}{1}$$

$$\lim_{x \rightarrow \frac{\pi}{3}} \left(\frac{\sin 2x}{\sin x} \right) = 1$$

$$\begin{aligned}
 5. \lim_{x \rightarrow 8} (2x + \sqrt[3]{x} - 4) \\
 &= 2(8) + \sqrt[3]{8} - 4 \\
 &= 16 + 2 - 4 \\
 \lim_{x \rightarrow 8} (2x + \sqrt[3]{x} - 4) &= 14
 \end{aligned}$$

$$\begin{aligned}
 6. \lim_{x \rightarrow 2} (4x - 3)(x^2 + 5) \\
 &= (4(2) - 3)((2)^2 + 5) \\
 &= (8 - 3)(4 + 5) \\
 &= (5)(9)
 \end{aligned}$$

$$\lim_{x \rightarrow 2} (4x - 3) = 5$$

$$\begin{aligned}
 7. \lim_{x \rightarrow 3} \left(\frac{\sqrt{3x}}{x\sqrt{x+1}} \right) \\
 &= \frac{\sqrt{3(3)}}{3\sqrt{3+1}} \\
 &= \frac{\sqrt{9}}{3\sqrt{4}}
 \end{aligned}$$

$$\lim_{x \rightarrow 3} \left(\frac{\sqrt{3x}}{x\sqrt{x+1}} \right) = \frac{1}{2}$$

$$\begin{aligned}
 8. \lim_{x \rightarrow 0} \left(\frac{3x+2}{x^2-2x+4} \right) \\
 &= \frac{3(0)+2}{0^2-2(0)+4} \\
 &= \frac{2}{4}
 \end{aligned}$$

$$\lim_{x \rightarrow 0} \left(\frac{3x+2}{x^2-2x+4} \right) = \frac{1}{2}$$

EXERCISE 1.3

Evaluate each of the following.

$$\begin{aligned}
 1. \lim_{x \rightarrow 4} \left(\frac{x^3 - 64}{x^2 - 16} \right) \\
 &= \frac{(x-4)(x^2+4x+16)}{(x-4)(x+4)} \\
 &= \frac{x^2+4x+16}{x+4} \\
 &= \frac{(4)^2+4(4)+16}{4+4} \\
 &= \frac{16+16+16}{8} \\
 &= \frac{48}{8}
 \end{aligned}$$

$$\lim_{x \rightarrow 4} \left(\frac{x^3 - 64}{x^2 - 16} \right) = 6$$

$$\begin{aligned}
 2. \lim_{x \rightarrow 2} \left(\frac{x^2+2x-8}{3x-6} \right) \\
 &= \frac{(x+4)(x-2)}{3(x-2)} \\
 &= \frac{x+4}{3} \\
 &= \frac{2+4}{3} \\
 &= \frac{6}{3}
 \end{aligned}$$

$$\lim_{x \rightarrow 2} \left(\frac{x^2+2x-8}{3x-6} \right) = 2$$

$$\begin{aligned}
 3. \lim_{x \rightarrow 3} \left(\frac{x^3 - 13x + 12}{x^3 - 14x + 15} \right) \\
 &= \frac{(x-3)(x^2+3x-4)}{(x-3)(x^2+3x-5)} \\
 &= \frac{x^2+3x-4}{x^2+3x-5} \\
 &= \frac{(3)^2+3(3)-4}{(3)^2+3(3)-5}
 \end{aligned}$$

$$= \frac{9+9-4}{9+9-5}$$

$$\lim_{x \rightarrow 3} \left(\frac{x^3 - 13x + 12}{x^3 - 14x + 15} \right) = \frac{14}{13}$$

$$4. \lim_{x \rightarrow 2} \left(\frac{x^3 - x^2 - x - 2}{2x^3 - 5x^2 + 5x - 6} \right)$$

$$= \frac{(x-2)(x^2+x+1)}{(x-2)(2x^2-x+3)}$$

$$= \frac{x^2+x+1}{2x^2-x+3}$$

$$= \frac{(2)^2+2+1}{2(2)^2-2+3}$$

$$= \frac{4+2+1}{8-2+3}$$

$$\lim_{x \rightarrow 2} \left(\frac{x^3 - x^2 - x - 2}{2x^3 - 5x^2 + 5x - 6} \right) = \frac{7}{9}$$

$$5. \lim_{x \rightarrow 0} \left(\frac{(x+3)^2 - 9}{2x} \right)$$

$$= \frac{(x+3)^2 - (3)^2}{2x}$$

$$= \frac{[(x+3)-3][(x+3)+3]}{2x}$$

$$= \frac{x(x+6)}{2x}$$

$$= \frac{x+6}{2}$$

$$= \frac{6}{2}$$

$$\lim_{x \rightarrow 0} \left(\frac{(x+3)^2 - 9}{2x} \right) = 3$$

$$6. \lim_{x \rightarrow 0} \left(\frac{\sqrt{x+16} - 4}{x} \right)$$

$$= \frac{\sqrt{x+16} - 4}{x} * \frac{\sqrt{x+16} + 4}{\sqrt{x+16} + 4}$$

$$= \frac{x+16-16}{x(\sqrt{x+16} + 4)}$$

$$= \frac{x}{x(\sqrt{x+16} + 4)}$$

$$= \frac{1}{\sqrt{x+16} + 4}$$

$$= \frac{1}{\sqrt{16} + 4}$$

$$= \frac{1}{4+4}$$

$$\lim_{x \rightarrow 0} \left(\frac{\sqrt{x+16} - 4}{x} \right) = \frac{1}{8}$$

$$7. \lim_{x \rightarrow 1} \left(\frac{x-1}{\sqrt{x+3} - 2} \right)$$

$$= \frac{x-1}{\sqrt{x+3} - 2} * \frac{\sqrt{x+3} + 2}{\sqrt{x+3} + 2}$$

$$= \frac{(x-1)(\sqrt{x+3} + 2)}{x+3-4}$$

$$= \frac{(x-1)(\sqrt{x+3} + 2)}{x-1}$$

$$= \sqrt{x+3} + 2$$

$$= \sqrt{1+3} + 2$$

$$= \sqrt{4} + 2$$

$$= 2+2$$

$$\lim_{x \rightarrow 1} \left(\frac{x-1}{\sqrt{x+3} - 2} \right) = 4$$

$$8. \lim_{x \rightarrow 8} \left(\frac{\sqrt[3]{x} - 2}{x-8} \right)$$

$$= \frac{\sqrt[3]{x} - 2}{x-8} * \frac{\sqrt[3]{x^2} + 2\sqrt[3]{x} + 4}{\sqrt[3]{x^2} + 2\sqrt[3]{x} + 4}$$

$$\begin{aligned}
&= \frac{x-8}{(x-8)(\sqrt[3]{x^2} + 2\sqrt[3]{x} + 4)} \\
&= \frac{1}{\sqrt[3]{x^2} + 2\sqrt[3]{x} + 4} \\
&= \frac{1}{\sqrt[3]{8^2} + 2\sqrt[3]{8} + 4} \\
&= \frac{1}{\sqrt[3]{64} + 2(2) + 4} \\
&= \frac{1}{4+4+4}
\end{aligned}$$

$$\lim_{x \rightarrow 8} \left(\frac{\sqrt[3]{x} - 2}{x - 8} \right) = \frac{1}{12}$$

$$9. \lim_{x \rightarrow 4} \left(\frac{\frac{1}{x} - \frac{1}{4}}{x - 4} \right)$$

$$= \frac{4-x}{4x}$$

$$= \frac{4-x}{4x(x-4)}$$

$$= -\frac{x-4}{4x(x-4)}$$

$$= -\frac{1}{4x}$$

$$= -\frac{1}{4(4)}$$

$$\lim_{x \rightarrow 4} \left(\frac{\frac{1}{x} - \frac{1}{4}}{x - 4} \right) = -\frac{1}{16}$$

$$10. \lim_{x \rightarrow 2} \left(\frac{x^3 - 8}{x^2 - 4} \right)$$

$$= \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$$

$$= \frac{x^2+2x+4}{x+2}$$

$$= \frac{(2)^2+2(2)+4}{2+2}$$

$$= \frac{4+4+4}{4}$$

$$= \frac{12}{4}$$

$$\lim_{x \rightarrow 2} \left(\frac{x^3 - 8}{x^2 - 4} \right) = 3$$

$$11. \lim_{x \rightarrow 3} \left(\frac{x-3}{\sqrt{x-2} - \sqrt{4-x}} \right)$$

$$= \frac{x-3}{\sqrt{x-2} - \sqrt{4-x}} \cdot \frac{\sqrt{x-2} + \sqrt{4-x}}{\sqrt{x-2} + \sqrt{4-x}}$$

$$= \frac{(x-3)(\sqrt{x-2} + \sqrt{4-x})}{(x-2) - (4-x)}$$

$$= \frac{(x-3)(\sqrt{x-2} + \sqrt{4-x})}{2x-6}$$

$$= \frac{(x-3)(\sqrt{x-2} + \sqrt{4-x})}{2(x-3)}$$

$$= \frac{\sqrt{x-2} + \sqrt{4-x}}{2}$$

$$= \frac{\sqrt{3-2} + \sqrt{4-3}}{2}$$

$$= \frac{1+1}{2}$$

$$= \frac{2}{2}$$

$$\lim_{x \rightarrow 3} \left(\frac{x-3}{\sqrt{x-2} - \sqrt{4-x}} \right) = 1$$

$$12. \lim_{x \rightarrow 0} \left[\left(\frac{1}{x} \right) \left(\frac{1}{3} - \frac{1}{\sqrt{x+9}} \right) \right]$$

$$= \left(\frac{1}{x} \right) \left(\frac{\sqrt{x+9} - 3}{3\sqrt{x+9}} \right)$$

$$= \left(\frac{1}{3x} \right) \left(\frac{\sqrt{x+9} - 3}{\sqrt{x+9}} \right) \left(\frac{\sqrt{x+9} + 3}{\sqrt{x+9} + 3} \right)$$

$$= \left(\frac{1}{3x} \right) \left(\frac{(x+9) - 9}{\sqrt{x+9}(\sqrt{x+9} + 3)} \right)$$

$$\begin{aligned}
&= \left(\frac{1}{3x} \right) \left(\frac{x}{(x+9)+3\sqrt{x+9}} \right) \\
&= \frac{x}{3x[(x+9)+3\sqrt{x+9}]} \\
&= \frac{1}{3(x+9)+9\sqrt{x+9}} \\
&= \frac{1}{3(0)+27+9\sqrt{0+9}} \\
&= \frac{1}{0+27+27}
\end{aligned}$$

$$\lim_{x \rightarrow 0} \left[\left(\frac{1}{x} \right) \left(\frac{1}{3} - \frac{1}{\sqrt{x+9}} \right) \right] = \frac{1}{54}$$

$$\begin{aligned}
13. \lim_{x \rightarrow 3} \left(\frac{\sqrt{x^2-9}}{x-3} \right) \\
&= \frac{\sqrt{x^2-9}}{x-3} * \frac{\sqrt{x^2-9}}{\sqrt{x^2-9}} \\
&= \frac{x^2-9}{(x-3)\sqrt{x^2-9}} \\
&= \frac{(x-3)\sqrt{x^2-9}}{(x-3)\sqrt{x^2-9}} \\
&= \frac{x+3}{\sqrt{x^2-9}} \\
&= \frac{3+3}{\sqrt{3^2-9}} \\
&= \frac{6}{0}
\end{aligned}$$

$$\lim_{x \rightarrow 3} \left(\frac{\sqrt{x^2-9}}{x-3} \right) = \infty \text{ The Limit does not exist}$$

$$\begin{aligned}
14. \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{\tan 2x}{\sec 2x} \right) \\
&= \frac{\sin 2x}{\cos 2x} \\
&= \frac{1}{\cos 2x} \\
&= \frac{\sin 2x \cos 2x}{\cos 2x}
\end{aligned}$$

$$\begin{aligned}
&= \sin 2x \\
&= \sin 2 \left(\frac{\pi}{4} \right) \\
&= \sin \frac{\pi}{2}
\end{aligned}$$

$$\lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{\tan 2x}{\sec 2x} \right) = 1$$

$$\begin{aligned}
15. \lim_{x \rightarrow 0} \left(\frac{\sin^3 x}{\sin x - \tan x} \right) \\
&= \frac{\sin^3 x}{\sin x - \frac{\sin x}{\cos x}} \\
&= \frac{\sin^3 x}{\sin x \cos x - \sin x} \\
&= \frac{\sin^3 x \cos x}{\sin x \cos x - \sin x} \\
&= \frac{\sin^3 x \cos x}{\sin x (\cos x - 1)} \\
&= \frac{\sin^2 x \cos x}{\cos x - 1} \\
&= \frac{(1 - \cos^2 x) \cos x}{\cos x - 1} \\
&= \frac{(1 - \cos x)(1 + \cos x) \cos x}{\cos x - 1}
\end{aligned}$$

$$\begin{aligned}
&= -(1 + \cos x) \cos x \\
&= -(1 + \cos 0) \cos 0
\end{aligned}$$

$$\begin{aligned}
&= -(1 + 1) \cdot 1 \\
\lim_{x \rightarrow 0} \left(\frac{\sin^3 x}{\sin x - \tan x} \right) &= -2
\end{aligned}$$

$$16. \lim_{x \rightarrow 0} \left(\frac{1 - \cos^2 x}{1 + \cos x} \right)$$

$$= \frac{(1 - \cos x)(1 + \cos x)}{1 + \cos x}$$

$$= 1 - \cos x$$

$$= 1 - \cos 0$$

$$= 1 - 1$$

$$\lim_{x \rightarrow 0} \left(\frac{1 - \cos^2 x}{1 + \cos x} \right) = 0$$

$$17. \lim_{x \rightarrow 0} \left(\frac{\sin x \sin 2x}{1 - \cos x} \right)$$

$$= \frac{\sin x (2 \sin x \cos x)}{1 - \cos x}$$

$$= \frac{2 \sin^2 x \cos x}{1 - \cos x}$$

$$= \frac{2 \cos x (1 - \cos^2 x)}{1 - \cos x}$$

$$= \frac{2 \cos x (1 - \cos x)(1 + \cos x)}{1 - \cos x}$$

$$= 2 \cos x (1 + \cos x)$$

$$= 2 \cos 0 (1 + \cos 0)$$

$$= 2(1)(1+1)$$

$$= 2(2)$$

$$\lim_{x \rightarrow 0} \left(\frac{\sin x \sin 2x}{1 - \cos x} \right) = 4$$

18.

$$= \frac{1 - \cos^2 x}{1 + \cos x}$$

$$= \frac{(1 - \cos x)(1 + \cos x)}{1 + \cos x}$$

$$= 1 - \cos x$$

$$= 1 - \cos \pi$$

$$= 1 - (-1)$$

$$\lim_{x \rightarrow \pi} \left(\frac{\sin^2 x}{1 + \cos x} \right) = 2$$

If $f(x) = \sqrt{x}$, find;

$$19. \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4}$$

$$= \frac{\sqrt{x} - \sqrt{4}}{x - 4}$$

$$= \frac{\sqrt{x} - 2}{x - 4}$$

$$= \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2}$$

$$= \frac{x - 4}{(x - 4)(\sqrt{x} + 2)}$$

$$= \frac{1}{\sqrt{x} + 2}$$

$$= \frac{1}{\sqrt{4} + 2}$$

$$= \frac{1}{2 + 2}$$

$$\lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} = \frac{1}{4}$$

$$20. \lim_{x \rightarrow 0} \left(\frac{f(9+x) - f(9)}{x} \right)$$

$$= \frac{\sqrt{9+x} - \sqrt{9}}{x}$$

$$= \frac{\sqrt{x+9} - 3}{x}$$

$$= \frac{\sqrt{x+9} - 3}{x} \cdot \frac{\sqrt{x+9} + 3}{\sqrt{x+9} + 3}$$

$$\begin{aligned}
&= \frac{x+9-9}{x(\sqrt{x+9}+3)} \\
&= \frac{x}{x(\sqrt{x+9}+3)} \\
&= \frac{1}{\sqrt{x+9}+3} \\
&= \frac{1}{\sqrt{0+9}+3} \\
&= \frac{1}{\sqrt{9}+3} \\
&= \frac{1}{3+3}
\end{aligned}$$

$$\lim_{x \rightarrow 0} \left(\frac{f(9+x) - f(9)}{x} \right) = \frac{1}{6}$$

If $f(x) = x^2 - 2x + 3$, find;

$$\begin{aligned}
21. \lim_{x \rightarrow 2} \left(\frac{f(x) - f(2)}{x - 2} \right) \\
&= \frac{(x^2 - 2x + 3) - ((2)^2 - 2(2) + 3)}{x - 2} \\
&= \frac{x^2 - 2x + 3 - 4 + 4 - 3}{x - 2} \\
&= \frac{x^2 - 2x}{x - 2} \\
&= \frac{x(x - 2)}{x - 2} \\
&= x
\end{aligned}$$

$$\lim_{x \rightarrow 2} \left(\frac{f(x) - f(2)}{x - 2} \right) = 2$$

$$\begin{aligned}
22. \lim_{x \rightarrow 0} \left(\frac{f(x+2) - f(2)}{x} \right) \\
&= \frac{[(x+2)^2 - 2(x+2) + 3] - ((2)^2 - 2(2) + 3)}{x} \\
&= \frac{[x^2 + 4x + 4 - 2x - 4 + 3] - 4 + 4 - 3}{x} \\
&= \frac{x^2 + 2x}{x} \\
&= \frac{x(x+2)}{x} \\
&= x + 2
\end{aligned}$$

$$= x + 2$$

$$= 0 + 2$$

$$\lim_{x \rightarrow 0} \left(\frac{f(x+2) - f(2)}{x} \right) = 2$$

EXERCISE 14

Evaluate each of the following.

$$1. \lim_{x \rightarrow \infty} \left(\frac{6x^3 + 4x^2 + 5}{8x^3 + 7x - 3} \right)$$

REMINDER:

$$\lim_{x \rightarrow \infty} \left(\frac{1}{x} \right) = 0$$

$$= \frac{6x^3 + 4x^2 + 5}{8x^3 + 7x - 3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \frac{\frac{6x^3}{x^3} + \frac{4x^2}{x^3} + \frac{5}{x^3}}{\frac{8x^3}{x^3} + \frac{7x}{x^3} - \frac{3}{x^3}}$$

$$= \frac{6 + \frac{4}{x} + \frac{5}{x^3}}{8 + \frac{7}{x^2} - \frac{3}{x^3}}$$

$$= \frac{6 + 0 + 0}{8 + 0 - 0}$$

$$= \frac{6}{8}$$

$$= \frac{6}{8}$$

$$= \frac{6}{8}$$

$$\lim_{x \rightarrow \infty} \left(\frac{6x^3 + 4x^2 + 5}{8x^3 + 7x - 3} \right) = \frac{3}{4}$$

$$2. \lim_{x \rightarrow \infty} \left(\frac{3x^2 + x + 2}{x^3 + 8x + 1} \right)$$

$$\begin{aligned}
&= \frac{3x^2+x+2}{x^3+8x+1} \cdot \frac{1}{x^3} \\
&= \frac{\frac{3x^2}{x^3} + \frac{x}{x^3} + \frac{2}{x^3}}{\frac{x^3}{x^3} + \frac{8x}{x^3} + \frac{1}{x^3}} \\
&= \frac{\frac{3}{x} + \frac{1}{x^2} + \frac{2}{x^3}}{1 + \frac{8}{x^2} + \frac{1}{x^3}} \\
&= \frac{0+0+0}{1+0+0}
\end{aligned}$$

$$\lim_{x \rightarrow \infty} \left(\frac{3x^2+x+2}{x^3+8x+1} \right) = 0$$

$$3. \lim_{x \rightarrow \infty} \left(\frac{4x+5}{x^2+1} \right)$$

$$\begin{aligned}
&= \frac{4x+5}{x^2+1} \cdot \frac{1}{x^2} \\
&= \frac{\frac{4x}{x^2} + \frac{5}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}} \\
&= \frac{\frac{4}{x} + \frac{5}{x^2}}{1 + \frac{1}{x^2}} \\
&= \frac{0+0}{1+0}
\end{aligned}$$

$$\lim_{x \rightarrow \infty} \left(\frac{4x+5}{x^2+1} \right) = 0$$

$$4. \lim_{x \rightarrow \infty} \left(\frac{x^3+x+2}{x^2-1} \right)$$

$$\begin{aligned}
&= \frac{x^3+x+2}{x^2-1} \cdot \frac{1}{x^3} \\
&= \frac{\frac{x^3}{x^3} + \frac{x}{x^3} + \frac{2}{x^3}}{\frac{x^2}{x^3} - \frac{1}{x^3}} \\
&= \frac{1 + \frac{1}{x^2} + \frac{2}{x^3}}{\frac{1}{x} - \frac{1}{x^3}} \\
&= \frac{1+0+0}{0-0}
\end{aligned}$$

$$\lim_{x \rightarrow \infty} \left(\frac{x^3+x+2}{x^2-1} \right) = \infty$$

$$5. \lim_{x \rightarrow \infty} \left(\frac{8x-5}{\sqrt{4x^2+3}} \right)$$

$$\begin{aligned}
&= \frac{8x-5}{\sqrt{4x^2+3}} \cdot \frac{1}{x} \\
&= \frac{\frac{8x}{x} - \frac{5}{x}}{\sqrt{\frac{4x^2}{x^2} + \frac{3}{x^2}}} \\
&= \frac{8 - \frac{5}{x}}{\sqrt{4 + \frac{3}{x^2}}} \\
&= \frac{8-0}{\sqrt{4+0}}
\end{aligned}$$

$$= \frac{8}{2}$$

$$\lim_{x \rightarrow \infty} \left(\frac{8x-5}{\sqrt{4x^2+3}} \right) = 4$$

$$\begin{aligned}
6. \lim_{x \rightarrow \infty} \left(\frac{x^3}{(2x-1)^2} \right) \\
&= \frac{x^3}{4x^2 - 4x + 1} \\
&= \frac{x^3}{4x^2 - 4x + 1} \cdot \frac{1}{x^3} \\
&= \frac{\frac{x^3}{x^3}}{\frac{4x^2}{x^3} - \frac{4x}{x^3} + \frac{1}{x^3}} \\
&= \frac{1}{\frac{4}{x} - \frac{4}{x^2} + \frac{1}{x^3}} \\
&= \frac{1}{0+0+0}
\end{aligned}$$

$$\lim_{x \rightarrow \infty} \left(\frac{x^3}{(2x-1)^2} \right) = \infty$$

$$\begin{aligned}
7. \lim_{x \rightarrow \infty} \left(\frac{(x+3)^3 - (x-2)^3}{x^2} \right) \\
&= \frac{(x^3 + 9x^2 + 27x + 27) - (x^3 - 6x^2 + 12x - 8)}{x^2} \\
&= \frac{x^3 + 9x^2 + 27x + 27 - x^3 + 6x^2 - 12x + 8}{x^2} \\
&= \frac{15x^2 + 15x + 35}{x^2} \\
&= \frac{15x^2 + 15x + 35}{x^2} \cdot \frac{1}{x^2} \\
&= \frac{\frac{15x^2}{x^2} + \frac{15x}{x^2} + \frac{35}{x^2}}{\frac{x^2}{x^2}} \\
&= \frac{15 + \frac{15}{x} + \frac{35}{x^2}}{1} \\
&= \frac{15+0+0}{1}
\end{aligned}$$

$$\lim_{x \rightarrow \infty} \left(\frac{(x+3)^3 - (x-2)^3}{x^2} \right) = 15$$

$$\begin{aligned}
8. \lim_{x \rightarrow \infty} \left(\frac{\sqrt{9x^2+4}}{6x+1} \right) \\
&= \frac{\sqrt{9x^2+4}}{6x+1} \cdot \frac{1}{x} \\
&= \frac{\sqrt{\frac{9x^2}{x^2} + \frac{4}{x^2}}}{\frac{6x}{x} + \frac{1}{x}} \\
&= \frac{\sqrt{9 + \frac{4}{x^2}}}{6 + \frac{1}{x}} \\
&= \frac{\sqrt{9+0}}{6+0} \\
&= \frac{3}{6}
\end{aligned}$$

$$\lim_{x \rightarrow \infty} \left(\frac{\sqrt{9x^2+4}}{6x+1} \right) = \frac{1}{2}$$

EXERCISE 1.5

Find the value or values of x for which the function is discontinuous. (The function is discontinuous if the denominator is equal to 0)

$$1. \frac{3x}{x-5}$$

$$x-5=0$$

$$x=5$$

Check

$$\frac{3x}{x-5}$$

$$= \frac{3(5)}{5-5}$$

$$= \frac{15}{0}, \text{ Therefore, the function is discontinuous if } x=5.$$

$$2. \frac{3x+2}{x^2-8x+15}$$

$$x^2-8x+15=0$$

$$(x-5)(x-3)=0$$

$$x=5, x=3$$

Check

$$\frac{3(5)+2}{(5)^2-8(5)+15} \quad \frac{3(3)+2}{(3)^2-8(3)+15}$$

$$= \frac{17}{0} \quad = \frac{11}{0}$$

Therefore, the function is discontinuous if $x=5$ and $x=3$.

$$3. \frac{5x+1}{x^2+4}$$

None. The function is continuous.

$$4. \frac{6x}{x^2-9}$$

$$x^2-9=0$$

$$(x-3)(x+3)=0$$

$$x=3, x=-3$$

Check

$$\frac{6x}{x^2-9}$$

$$= \frac{6(\pm 3)}{(\pm 3)^2-9}$$

$$= \frac{\pm 18}{0}$$

Therefore, the function is discontinuous if $x = \pm 3$.

$$5. \frac{1}{2^x-8}$$

$$2^x-8=0$$

$$2^x=8$$

$\log_2 8 = x$, Use calculator or just simply assume a number that makes the denominator zero.

$$x=3 \quad \log_2 8 = x$$

Check

$$\frac{1}{2^x-8} \quad \log_2 2^3 = x, \log_a a^r = r$$

$$= \frac{1}{2^3-8} \quad \log_2 2^3 = 3$$

$$= \frac{1}{8-8}$$

$$= \frac{1}{0}, \text{ Therefore, the function is discontinuous if } x=3.$$

$$6. \frac{x+3}{x^3-3x^2+2x}$$

$$x^3-3x^2+2x=0$$

$$x(x^2-3x+2)=0$$

$$x(x-2)(x-1)=0$$

$$x=0, x=2, x=1$$

Check

$$\frac{x+3}{x^3-3x^2+2x}$$

$$= \frac{0+3}{(0)^3-3(0)^2+2(0)}$$

$$f(0) = \frac{3}{0}$$

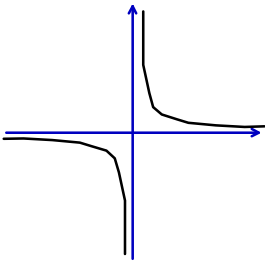
$$f(2) = \frac{5}{0}$$

$$f(1) = \frac{4}{0}$$

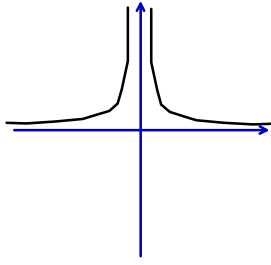
EXERCISE 1.6

Sketch the graph of the following functions:

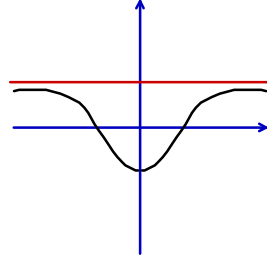
$$1. y = \frac{4}{x}$$



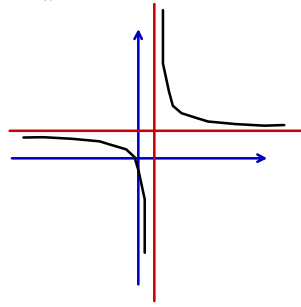
$$2. y = \frac{6}{x^2}$$



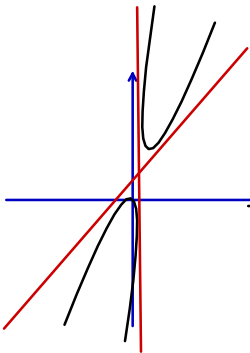
$$3. y = \frac{x^2-1}{x^2+1}$$



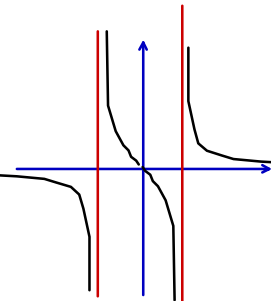
$$4. y = \frac{2x}{x-1}$$



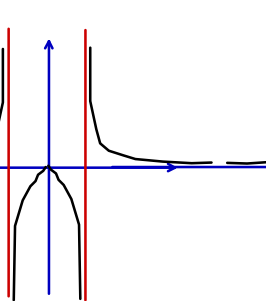
$$5. y = \frac{x^2}{x-2}$$



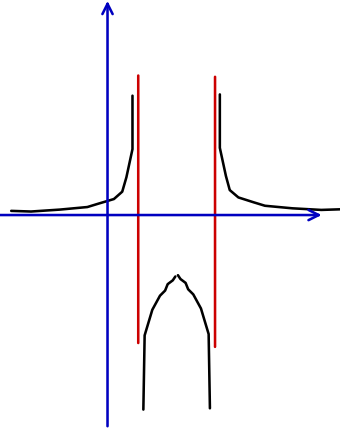
$$6. y = \frac{2x}{x^2-4}$$



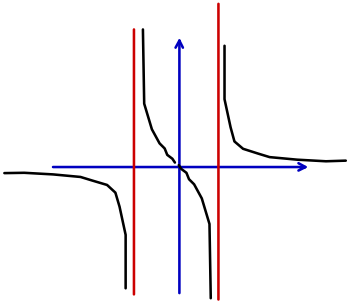
$$7. y = \frac{2}{x^2-1}$$



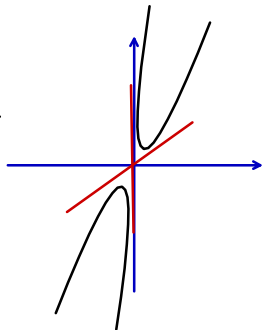
$$8. y = \frac{x-2}{x^2-9x+20}$$



$$9. y = \frac{4x}{x^2-9}$$



$$10. y = \frac{x^2+1}{x}$$



CHAPTER 2: DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

EXERCISE 2.1

Find the derivative by use of Delta Method.

$$1. y = 4x^2 - 5x$$

$$y + \Delta y = 4(x + \Delta x)^2 - 5(x + \Delta x)$$

$$\Delta y = [4(x + \Delta x)^2 - 5(x + \Delta x)] - (4x^2 - 5x)$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$= \frac{[4(x + \Delta x)^2 - 5(x + \Delta x)] - (4x^2 - 5x)}{\Delta x}$$

$$= \frac{[4(x^2 + 2x\Delta x + \Delta x^2) - 5x - 5\Delta x] - 4x^2 + 5x}{\Delta x}$$

$$= \frac{4x^2 + 8x\Delta x + 4\Delta x^2 - 5x - 5\Delta x - 4x^2 + 5x}{\Delta x}$$

$$= \frac{8x\Delta x + 4\Delta x^2 - 5\Delta x}{\Delta x}$$

$$= \frac{\Delta x(8x + 4\Delta x - 5)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} (8x + 4\Delta x - 5)$$

$$= \lim_{\Delta x \rightarrow 0} (8x + 4(0) - 5)$$

$$\frac{dy}{dx} = 8x - 5$$

$$2. y = x^3 + 2x$$

$$y + \Delta y = (x + \Delta x)^3 + 2(x + \Delta x)$$

$$\Delta y = [(x + \Delta x)^3 + 2(x + \Delta x)] - (x^3 + 2x)$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$= \frac{[(x + \Delta x)^3 + 2(x + \Delta x)] - (x^3 + 2x)}{\Delta x}$$

$$= \frac{[(x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) + 2x + 2\Delta x] - x^3 - 2x}{\Delta x}$$

$$= \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x}$$

$$= \frac{3x^2\Delta x + 2x\Delta x^2 + \Delta x^3 + 2\Delta x}{\Delta x}$$

$$= \frac{\Delta x(3x^2 + 2x\Delta x + \Delta x^2 + 2)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} (3x^2 + 2x\Delta x + \Delta x^2 + 2)$$

$$= \lim_{\Delta x \rightarrow 0} (3x^2 + 2x(0) + (0)^2 + 2)$$

$$\frac{dy}{dx} = 3x^2 + 2$$

$$3. y = 4\sqrt{x}$$

$$y + \Delta y = 4\sqrt{x + \Delta x}$$

$$\Delta y = 4\sqrt{x + \Delta x} - 4\sqrt{x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$= \frac{4\sqrt{x + \Delta x} - 4\sqrt{x}}{\Delta x}$$

$$= \frac{4\sqrt{x + \Delta x} - 4\sqrt{x}}{\Delta x} \cdot \frac{4\sqrt{x + \Delta x} + 4\sqrt{x}}{4\sqrt{x + \Delta x} + 4\sqrt{x}}$$

$$= \frac{16(x + \Delta x) - 16x}{\Delta x(4\sqrt{x + \Delta x} + 4\sqrt{x})}$$

$$= \frac{16x + 16\Delta x - 16x}{\Delta x(4\sqrt{x + \Delta x} + 4\sqrt{x})}$$

$$= \frac{16\Delta x}{\Delta x(4\sqrt{x + \Delta x} + 4\sqrt{x})}$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{16}{4\sqrt{x + \Delta x} + 4\sqrt{x}} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{16}{4\sqrt{x+0} + 4\sqrt{x}} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{16}{8\sqrt{x}} \right)$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{x}}$$

$$4. y = \frac{6}{x}$$

$$y + \Delta y = \frac{6}{x + \Delta x}$$

$$\Delta y = \frac{6}{x + \Delta x} - \frac{6}{x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$= \frac{\frac{6}{x + \Delta x} - \frac{6}{x}}{\Delta x}$$

$$= \frac{\frac{6x - 6(x + \Delta x)}{x(x + \Delta x)}}{\Delta x}$$

$$= \frac{6x - 6x - 6\Delta x}{x\Delta x(x + \Delta x)}$$

$$= \frac{-6\Delta x}{x\Delta x(x + \Delta x)}$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{-6}{x(x + \Delta x)} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{-6}{x(x + 0)} \right)$$

$$\frac{dy}{dx} = -\frac{6}{x^2}$$

$$5. y = \sqrt[3]{x}$$

$$y + \Delta y = \sqrt[3]{x + \Delta x}$$

$$\Delta y = \sqrt[3]{x + \Delta x} - \sqrt[3]{x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$= \frac{\sqrt[3]{x + \Delta x} - \sqrt[3]{x}}{\Delta x}$$

$$= \frac{\Delta x}{\sqrt[3]{x + \Delta x} - \sqrt[3]{x}} \cdot \frac{\sqrt[3]{(x + \Delta x)^2} + \sqrt[3]{x(x + \Delta x)} + \sqrt[3]{x^2}}{\sqrt[3]{(x + \Delta x)^2} + \sqrt[3]{x(x + \Delta x)} + \sqrt[3]{x^2}}$$

$$= \frac{\Delta x \left(\sqrt[3]{(x + \Delta x)^2} + \sqrt[3]{x(x + \Delta x)} + \sqrt[3]{x^2} \right)}{\Delta x \left(\sqrt[3]{(x + \Delta x)^2} + \sqrt[3]{x(x + \Delta x)} + \sqrt[3]{x^2} \right)}$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{1}{\sqrt[3]{(x + \Delta x)^2} + \sqrt[3]{x(x + \Delta x)} + \sqrt[3]{x^2}} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{1}{\sqrt[3]{(x + 0)^2} + \sqrt[3]{x(x + 0)} + \sqrt[3]{x^2}} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{1}{\sqrt[3]{x^2} + \sqrt[3]{x^2} + \sqrt[3]{x^2}} \right)$$

$$\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}$$

$$6. y = 2 - 5x$$

$$y + \Delta y = 2 - 5(x + \Delta x)$$

$$\Delta y = [2 - 5(x + \Delta x)] - (2 - 5x)$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$= \frac{[2 - 5(x + \Delta x)] - (2 - 5x)}{\Delta x}$$

$$= \frac{2 - 5x - 5\Delta x - 2 + 5x}{\Delta x}$$

$$= \frac{-5\Delta x}{\Delta x}$$

$$\frac{dy}{dx} = -5$$

$$7. y = \sqrt{4x+3}$$

$$y + \Delta y = \sqrt{4(x + \Delta x) + 3}$$

$$\Delta y = \sqrt{4(x + \Delta x) + 3} - \sqrt{4x + 3}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$= \frac{\sqrt{4(x + \Delta x) + 3} - \sqrt{4x + 3}}{\Delta x} \cdot \frac{\sqrt{4(x + \Delta x) + 3} + \sqrt{4x + 3}}{\sqrt{4(x + \Delta x) + 3} + \sqrt{4x + 3}}$$

$$= \frac{4x + 4\Delta x + 3 - 4x - 3}{\Delta x(\sqrt{4(x + \Delta x) + 3} + \sqrt{4x + 3})}$$

$$= \frac{4\Delta x}{\Delta x(\sqrt{4(x + \Delta x) + 3} + \sqrt{4x + 3})}$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{4}{\sqrt{4(x + \Delta x) + 3} + \sqrt{4x + 3}} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{4}{\sqrt{4(x + 0) + 3} + \sqrt{4x + 3}} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{4}{2\sqrt{4x + 3}} \right)$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{4x + 3}}$$

$$8. y = \frac{2x}{x + 1}$$

$$y + \Delta y = \frac{2(x + \Delta x)}{(x + \Delta x) + 1}$$

$$\Delta y = \frac{2(x + \Delta x)}{(x + \Delta x) + 1} - \frac{2x}{x + 1}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$= \frac{\frac{2(x + \Delta x)}{(x + \Delta x) + 1} - \frac{2x}{x + 1}}{\Delta x}$$

$$= \frac{2(x + \Delta x)(x + 1) - 2x[(x + \Delta x) + 1]}{\Delta x[(x + \Delta x) + 1](x + 1)}$$

$$= \frac{2(x^2 + x + x\Delta x + \Delta x) - 2x^2 - 2x\Delta x - 2x}{\Delta x[(x + \Delta x) + 1](x + 1)}$$

$$= \frac{2x^2 + 2x + 2x\Delta x + 2\Delta x - 2x^2 - 2x\Delta x - 2x}{\Delta x[(x + \Delta x) + 1](x + 1)}$$

$$= \frac{2\Delta x}{\Delta x[(x + \Delta x) + 1](x + 1)}$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{2}{[(x + \Delta x) + 1](x + 1)} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{2}{[(x + 0) + 1](x + 1)} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{2}{(x + 1)(x + 1)} \right)$$

$$\frac{dy}{dx} = \frac{2}{(x + 1)^2}$$

$$9. y = \frac{3}{\sqrt{2x + 1}}$$

$$y + \Delta y = \frac{3}{\sqrt{2(x + \Delta x) + 1}}$$

$$\Delta y = \frac{3}{\sqrt{2(x + \Delta x) + 1}} - \frac{3}{\sqrt{2x + 1}}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$= \frac{\frac{3}{\sqrt{2(x + \Delta x) + 1}} - \frac{3}{\sqrt{2x + 1}}}{\Delta x}$$

$$= \frac{3\sqrt{2x + 1} - 3\sqrt{2(x + \Delta x) + 1}}{\Delta x(\sqrt{2x + 1})(\sqrt{2(x + \Delta x) + 1})}$$

$$= \frac{3\sqrt{2x + 1} - 3\sqrt{2(x + \Delta x) + 1}}{\Delta x(\sqrt{2x + 1})(\sqrt{2(x + \Delta x) + 1})} \cdot \frac{3\sqrt{2x + 1} + 3\sqrt{2(x + \Delta x) + 1}}{3\sqrt{2x + 1} + 3\sqrt{2(x + \Delta x) + 1}}$$

$$= \frac{9(2x + 1) - 9[2(x + \Delta x) + 1]}{\Delta x(\sqrt{2x + 1})(\sqrt{2(x + \Delta x) + 1})(3\sqrt{2x + 1} + 3\sqrt{2(x + \Delta x) + 1})}$$

$$= \frac{18x + 9 - 18x - 18\Delta x - 9}{\Delta x(\sqrt{2x + 1})(\sqrt{2(x + \Delta x) + 1})(3\sqrt{2x + 1} + 3\sqrt{2(x + \Delta x) + 1})}$$

$$= \frac{-18\Delta x}{\Delta x(\sqrt{2x + 1})(\sqrt{2(x + \Delta x) + 1})(3\sqrt{2x + 1} + 3\sqrt{2(x + \Delta x) + 1})}$$

$$= \frac{-18\Delta x}{\Delta x(\sqrt{2x + 1})(\sqrt{2(x + \Delta x) + 1})(3\sqrt{2x + 1} + 3\sqrt{2(x + \Delta x) + 1})}$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{-18}{(\sqrt{2x+1})(\sqrt{2(x+\Delta x)+1})(3\sqrt{2x+1}+3\sqrt{2(x+\Delta x)+1})} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{-18}{(\sqrt{2x+1})(\sqrt{2(x+0)+1})(3\sqrt{2x+1}+3\sqrt{2(x+0)+1})} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{-18}{(\sqrt{2x+1})(\sqrt{2x+1})(6\sqrt{2x+1})} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{-18}{6\sqrt{2x+1}(2x+1)} \right)$$

$$\frac{dy}{dx} = -\frac{3}{\sqrt{2x+1}(2x+1)}$$

$$10. y = \frac{5x^2}{4x-1}$$

$$y + \Delta y = \frac{5(x+\Delta x)^2}{4(x+\Delta x)-1}$$

$$\Delta y = \frac{5(x+\Delta x)^2}{4(x+\Delta x)-1} - \frac{5x^2}{4x-1}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$= \frac{5(x+\Delta x)^2}{4(x+\Delta x)-1} - \frac{5x^2}{4x-1}$$

$$= \frac{\Delta x}{[5(x+\Delta x)^2](4x-1) - (5x^2)[4(x+\Delta x)-1]}$$

$$= \frac{[(5x^2 + 10x\Delta x + 5\Delta x^2)](4x-1) - (5x^2)[4x+4\Delta x-1]}{\Delta x[4(x+\Delta x)-1](4x-1)}$$

$$= \frac{20x^3 - 5x^2 + 40x^2\Delta x - 10x\Delta x + 20x\Delta x^2 - 5\Delta x^2 - 20x^3 - 20x^2\Delta x + 5x^2}{\Delta x[4(x+\Delta x)-1](4x-1)}$$

$$= \frac{20x^2\Delta x - 10x\Delta x + 20x\Delta x^2 - 5\Delta x^2}{\Delta x[4(x+\Delta x)-1](4x-1)}$$

$$= \frac{\Delta x(20x^2 - 10x + 20x\Delta x - 5\Delta x)}{\Delta x[4(x+\Delta x)-1](4x-1)}$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{(20x^2 - 10x + 20x\Delta x - 5\Delta x)}{[4(x+\Delta x)-1](4x-1)} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{(20x^2 - 10x + 20x(0) - 5(0))}{[4(x+0)-1](4x-1)} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{(20x^2 - 10x)}{4x-1} \right)$$

$$\frac{dy}{dx} = \frac{10x(2x-1)}{(4x-1)^2}$$

$$11 \text{ Given } s = \sqrt{t}, \text{ find } \frac{ds}{dt}$$

$$s + \Delta s = \sqrt{t + \Delta t}$$

$$\Delta s = \sqrt{t + \Delta t} - \sqrt{t}$$

$$\frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \left(\frac{\Delta s}{\Delta t} \right)$$

$$= \frac{\sqrt{t + \Delta t} - \sqrt{t}}{\Delta t}$$

$$= \frac{\Delta t}{\Delta t} \cdot \frac{\sqrt{t + \Delta t} + \sqrt{t}}{\sqrt{t + \Delta t} + \sqrt{t}}$$

$$= \frac{t + \Delta t - t}{\Delta t(\sqrt{t + \Delta t} + \sqrt{t})}$$

$$= \frac{\Delta t}{\Delta t(\sqrt{t + \Delta t} + \sqrt{t})}$$

$$= \lim_{\Delta t \rightarrow 0} \left(\frac{1}{(\sqrt{t + \Delta t} + \sqrt{t})} \right)$$

$$= \lim_{\Delta t \rightarrow 0} \left(\frac{1}{(\sqrt{t+0} + \sqrt{t})} \right)$$

$$\frac{ds}{dt} = \frac{1}{2\sqrt{t}}$$

$$12 \text{ Given } A = \pi r^2, \text{ find } \frac{dA}{dr}$$

$$A + \Delta A = \pi(r + \Delta r)^2$$

$$\Delta A = \pi(r + \Delta r)^2 - \pi r^2$$

$$\frac{dA}{dr} = \lim_{\Delta r \rightarrow 0} \left(\frac{\Delta A}{\Delta r} \right)$$

$$\begin{aligned}
&= \frac{\pi(r+\Delta r)^2 - \pi r^2}{\Delta r} \\
&= \frac{\pi(r^2 + 2r\Delta r + \Delta r^2) - \pi r^2}{\Delta r} \\
&= \frac{\pi r^2 + 2\pi r\Delta r + \pi\Delta r^2 - \pi r^2}{\Delta r} \\
&= \frac{2\pi r\Delta r + \pi\Delta r^2}{\Delta r} \\
&= \frac{\Delta r(2\pi r + \pi\Delta r)}{\Delta r}
\end{aligned}$$

$$= \lim_{\Delta r \rightarrow 0} (2\pi r + \pi\Delta r)$$

$$= \lim_{\Delta r \rightarrow 0} (2\pi r + \pi(0))$$

$$\frac{dA}{dr} = 2\pi r$$

13. Given $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dr}$

$$V + \Delta V = \frac{4\pi(r+\Delta r)^3}{3}$$

$$\Delta V = \frac{4\pi(r+\Delta r)^3}{3} - \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = \lim_{\Delta r \rightarrow 0} \left(\frac{\Delta V}{\Delta r} \right)$$

$$= \frac{\frac{4\pi(r+\Delta r)^3}{3} - \frac{4\pi r^3}{3}}{\Delta r}$$

$$= \frac{12\pi(r+\Delta r)^3 - 12\pi r^3}{9\Delta r}$$

$$= \frac{12\pi(r^3 + 3r^2\Delta r + 3r\Delta r^2 + \Delta r^3) - 12\pi r^3}{9\Delta r}$$

$$= \frac{12\pi r^3 + 36\pi r^2\Delta r + 36\pi r\Delta r^2 + 12\pi\Delta r^3 - 12\pi r^3}{9\Delta r}$$

$$= \frac{3\Delta r(12\pi r^2 + 12\pi r\Delta r + 4\Delta r^2)}{9\Delta r}$$

$$= \lim_{\Delta r \rightarrow 0} \left(\frac{12\pi r^2 + 12\pi r\Delta r + 4\Delta r^2}{3} \right)$$

$$= \lim_{\Delta r \rightarrow 0} \left(\frac{12\pi r^2 + 12\pi r(0) + 4(0)}{3} \right)$$

$$= \lim_{\Delta r \rightarrow 0} \left(\frac{12\pi r^2}{3} \right)$$

$$\frac{dV}{dr} = 4\pi r^2$$

14. Given $S = 4\pi r^2$, find $\frac{dS}{dr}$

$$S + \Delta S = 4\pi(r+\Delta r)^2$$

$$\Delta S = 4\pi(r+\Delta r)^2 - 4\pi r^2$$

$$\frac{dS}{dr} = \lim_{\Delta r \rightarrow 0} \left(\frac{\Delta S}{\Delta r} \right)$$

$$= \frac{4\pi(r+\Delta r)^2 - 4\pi r^2}{\Delta r}$$

$$= \frac{4\pi(r^2 + 2r\Delta r + \Delta r^2) - 4\pi r^2}{\Delta r}$$

$$= \frac{4\pi r^2 + 8\pi r\Delta r + 4\pi\Delta r^2 - 4\pi r^2}{\Delta r}$$

$$= \frac{\Delta r(8\pi r + 4\pi\Delta r)}{\Delta r}$$

$$= \lim_{\Delta r \rightarrow 0} (8\pi r + 4\pi\Delta r)$$

$$= \lim_{\Delta r \rightarrow 0} (8\pi r + 4\pi(0))$$

$$\frac{dS}{dr} = 8\pi r$$

15. Given $S = \frac{2t+3}{3t-4}$, find $\frac{dS}{dt}$

$$S + \Delta S = \frac{2(t+\Delta t)+3}{3(t+\Delta t)-4}$$

$$\Delta S = \frac{2(t+\Delta t)+3}{3(t+\Delta t)-4} - \frac{2t+3}{3t-4}$$

$$\begin{aligned}
\frac{dS}{dt} &= \lim_{\Delta t \rightarrow 0} \left(\frac{\Delta S}{\Delta t} \right) \\
&= \frac{2(t+\Delta t)+3}{3(t+\Delta t)-4} - \frac{2t+3}{3t-4} \\
&= \frac{\Delta t}{[2(t+\Delta t)+3](3t-4) - (2t+3)[3(t+\Delta t)-4]} \\
&= \frac{\Delta t [3(t+\Delta t)-4](3t-4)}{(2t+2\Delta t+3)(3t-4) - (2t+3)(3t+3\Delta t-4)} \\
&= \frac{6t^2 - 8t + 6t\Delta t - 8\Delta t + 9t - 12 - 6t^2 - 9t - 6t\Delta t - 9\Delta t + 8t + 12}{\Delta t [3(t+\Delta t)-4](3t-4)} \\
&= \frac{-17\Delta t}{\Delta t [3(t+\Delta t)-4](3t-4)} \\
&= \lim_{\Delta t \rightarrow 0} \left(\frac{-17}{[3(t+\Delta t)-4](3t-4)} \right) \\
&= \lim_{\Delta t \rightarrow 0} \left(\frac{-17}{[3(t+0)-4](3t-4)} \right) \\
&= \lim_{\Delta t \rightarrow 0} \left(\frac{-17}{(3t-4)(3t-4)} \right) \\
\frac{dS}{dt} &= -\frac{17}{(3t-4)^2}
\end{aligned}$$

EXERCISE 2.2

Find $\frac{dy}{dx}$ of each of the following;

$$1. y = 5x^3 - 4x^2 + 3x - 6$$

Use power rule where; $\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$ and the constant rule where; $\frac{d}{dx}(c) = 0$

$$\frac{dy}{dx} = 5(3)x^{3-1} - 4(2)x^{2-1} + 3(1)x^{1-1} - 0$$

$$\frac{dy}{dx} = 15x^2 - 8x + 3$$

$$2. y = \sqrt[3]{x} + \frac{4}{x} + \sqrt{x}$$

Simplify $y = \sqrt[3]{x} + 4x^{-1} + \sqrt{x}$

Use power rule where; $\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$ or $\frac{d}{dx}(u^{\frac{n}{m}}) = \left(\frac{n}{m}\right) \left(\frac{du}{u^{\frac{n}{m}-1}}\right)$. du means the derivative of u.

$$\frac{dy}{dx} = \frac{1}{3x^{\frac{1}{3}-1}} + 4(-1)x^{-1-1} + \frac{1}{2x^{\frac{1}{2}-1}}$$

$$\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}} - \frac{4}{x^2} + \frac{1}{2\sqrt{x}}$$

$$3. y = \sqrt{5-6x}$$

Use power rule where; $\frac{d}{dx}(\sqrt{u}) = \frac{du}{2\sqrt{u}}$

$$u = 5 - 6x$$

$$du = -6dx$$

$$\frac{dy}{dx} = -\frac{6}{2\sqrt{5-6x}}$$

$$\frac{dy}{dx} = \frac{-3}{\sqrt{5-6x}}$$

$$4. y = \sqrt[3]{2x-7}$$

Use power rule where; $\frac{d}{dx}(u^{\frac{n}{m}}) = \left(\frac{n}{m}\right) \left(\frac{du}{u^{\frac{n}{m}-1}}\right)$.

$$u = 2x - 7$$

$$du = 2dx$$

$$\frac{n}{m} = \frac{1}{3}$$

$$\frac{dy}{dx} = \frac{2}{3\sqrt[3]{(2x-7)^2}}$$

$$5. y = (3x^2 - 4x + 1)^5$$

Use chain rule where; $\frac{d}{dx} ((u)^n) = n du(u)^{n-1}$

$$u = 3x^2 - 4x + 1$$

$$du = (6x - 4) dx$$

$$\frac{dy}{dx} = 5(6x - 4)(3x^2 - 4x + 1)^4$$

$$\frac{dy}{dx} = 30x - 20(3x^2 - 4x + 1)^4$$

$$6. y = \sqrt{7 + \sqrt{3x+1}}$$

Use chain rule where; $\frac{d}{dx} ((u)^n) = n du(u)^{n-1}$ and power rule where; $\frac{d}{dx} (\sqrt{u}) = \frac{du}{2\sqrt{u}}$

$$u = 7 + \sqrt{3x+1}$$

$$du = \frac{3}{2\sqrt{3x+1}} dx$$

$$\frac{dy}{dx} = \frac{3}{2\sqrt{7+\sqrt{3x+1}}}$$

$$\frac{dy}{dx} = \frac{3}{(2\sqrt{7+\sqrt{3x+1}})(2\sqrt{3x+1})}$$

$$\frac{dy}{dx} = \frac{3}{4\sqrt{3x+1}\sqrt{7+\sqrt{3x+1}}}$$

$$7. y = \frac{4x-5}{2x+1}$$

Use quotient rule where; $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{vdu - u dv}{v^2}$.

$$u = 4x - 5$$

$$du = 4 dx$$

$$v = 2x + 1$$

$$dv = 2 dx$$

$$\frac{dy}{dx} = \frac{(2x+1) \cdot 4 - (4x-5) \cdot 2}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{8x+4-8x+10}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{14}{(2x+1)^2}$$

$$8. y = \frac{3x+1}{\sqrt{3x^2+2}}$$

Use quotient rule where; $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{vdu - u dv}{v^2}$ and power rule where; $\frac{d}{dx} (\sqrt{u}) = \frac{du}{2\sqrt{u}}$

$$u = 3x + 1$$

$$du = 3 dx$$

$$v = \sqrt{3x^2+2}$$

$$dv = \frac{3x}{\sqrt{3x^2+2}} dx$$

$$\frac{dy}{dx} = \frac{\sqrt{3x^2+2}(3) - (3x+1)\left(\frac{3x}{\sqrt{3x^2+2}}\right)}{(3x^2+2)^2}$$

$$\frac{dy}{dx} = \frac{3\sqrt{3x^2+2} - \frac{3x^2+3x}{\sqrt{3x^2+2}}}{(3x^2+2)^2}$$

$$\frac{dy}{dx} = \frac{3\sqrt{3x^2+2}\sqrt{3x^2+2} - (9x^2+3x)}{(3x^2+2)\sqrt{3x^2+2}}$$

$$\frac{dy}{dx} = \frac{3(3x^2+2) - (9x^2+3x)}{(3x^2+2)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{9x^2+6-9x^2-3x}{(3x^2+2)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{6-3x}{\sqrt{(3x^2+2)^3}}$$

$$9. y = (2x+5)\sqrt{4x-1}$$

Use product rule where; $\frac{d}{dx}(uv) = u dv + v du$ and power rule where; $\frac{d}{dx}(\sqrt{u}) = \frac{du}{2\sqrt{u}}$

$$u = 2x+5$$

$$du = 2 dx$$

$$v = \sqrt{4x-1}$$

$$dv = \frac{2}{\sqrt{4x-1}}$$

$$\frac{dy}{dx} = (2x+5)\left(\frac{2}{\sqrt{4x-1}}\right) + 2\sqrt{4x-1}$$

$$\frac{dy}{dx} = \frac{2(2x+5)}{\sqrt{4x-1}} + 2\sqrt{4x-1}$$

$$\frac{dy}{dx} = \frac{2(2x+5) + 2(4x-1)}{\sqrt{4x-1}}$$

$$\frac{dy}{dx} = \frac{4x+10+8x-2}{\sqrt{4x-1}}$$

$$\frac{dy}{dx} = \frac{12x+8}{\sqrt{4x-1}}$$

$$10. y = (3x+4)^2(x-5)^3$$

Use chain rule where; $\frac{d}{dx}((u)^n) = n du(u)^{n-1}$ and product rule where; $\frac{d}{dx}(uv) = u dv + v du$

$$u = (3x+4)^2$$

$$du = 6(3x+4)$$

$$v = (x-5)^3$$

$$dv = 3(x-5)^2$$

$$\frac{dy}{dx} = (3x+4)^2(3(x-5)^2) + (x-5)^3(6(3x+4))$$

$$\frac{dy}{dx} = 3(x-5)^2[(3x+4)^2 + 2(x-5)(3x+4)]$$

$$\frac{dy}{dx} = 3(x-5)^2(3x+4)[(3x+4) + (2x-10)]$$

$$\frac{dy}{dx} = 3(x-5)^2(3x+4)(5x-6)$$

$$11. y = \left(\frac{2x-3}{5x+1}\right)^4$$

Use chain rule where; $\frac{d}{dx}((u)^n) = n du(u)^{n-1}$ and quotient rule where; $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v du - u dv}{v^2}$

$$u = 2x-3$$

$$du = 2 dx$$

$$v = 5x+1$$

$$dv = 5 dx$$

$$\frac{dy}{dx} = 4\left(\frac{2(5x+1) - 5(2x-3)}{(5x+1)^2}\right)\left(\frac{2x-3}{5x+1}\right)^3$$

$$\frac{dy}{dx} = 4\left(\frac{10x+2-10x+15}{(5x+1)^2}\right)\left(\frac{2x-3}{5x+1}\right)^3$$

$$\frac{dy}{dx} = 4\left(\frac{17}{(5x+1)^2}\right)\left(\frac{2x-3}{5x+1}\right)^3$$

$$\frac{dy}{dx} = \left(\frac{68}{(5x+1)^2}\right)\left(\frac{2x-3}{5x+1}\right)^3$$

$$\frac{dy}{dx} = \frac{68(2x-3)^3}{(5x+1)^5}$$

$$12. y = \frac{3x-4}{\sqrt{2x+5}}$$

Use quotient rule where; $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v du - u dv}{v^2}$, and power rule where; $\frac{d}{dx}(\sqrt{u}) = \frac{du}{2\sqrt{u}}$

$$u = 3x-4$$

$$du = 3 dx$$

$$v = \sqrt{2x+5}$$

$$dv = \frac{1}{\sqrt{2x+5}} dx$$

$$\frac{dy}{dx} = \frac{3\sqrt{2x+5} - \frac{3x-4}{\sqrt{2x+5}}}{((\sqrt{2x+5})^2)}$$

$$\frac{dy}{dx} = \frac{3(2x+5) - (3x-4)}{\sqrt{(2x+5)^3}}$$

$$\frac{dy}{dx} = \frac{6x+15-3x+4}{\sqrt{(2x+5)^3}}$$

$$\frac{dy}{dx} = \frac{3x+19}{\sqrt{(2x+5)^3}}$$

$$13. y = \sqrt[3]{\frac{x-6}{3x+4}}$$

Use chain rule where; $\frac{d}{dx}((u)^n) = n du(u)^{n-1}$ and quotient rule where; $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vdu - u dv}{v^2}$

$$u = x - 6$$

$$du = dx$$

$$v = 3x + 4$$

$$dv = 3 dx$$

$$\frac{dy}{dx} = \left(\frac{1}{3}\right) \left(\frac{(3x+4) - 3(x-6)}{(3x+4)^2}\right) \left(\frac{x-6}{3x+4}\right)^{-\frac{2}{3}}$$

$$\frac{dy}{dx} = \frac{3x+4 - 3x+18}{3(3x+4)^2 \sqrt[3]{\left(\frac{x-6}{3x+4}\right)^2}}$$

$$\frac{dy}{dx} = \frac{22}{3(3x+4)^2 \sqrt[3]{\left(\frac{x-6}{3x+4}\right)^2}}$$

$$\frac{dy}{dx} = \frac{22}{3(3x+4)^2 \frac{\sqrt[3]{(x-6)^2}}{\sqrt[3]{(3x+4)^2}}}$$

$$\frac{dy}{dx} = \frac{22}{3 \sqrt[3]{(x-6)^2} \sqrt[3]{(3x+4)^4}}$$

$$14. y = \sqrt[3]{x^2} - 4x^{-3}$$

Use power rule where; $\frac{d}{dx}(u^n) = nu^{n-1}$

$$y = x^{\frac{2}{3}} - 4x^{-3}$$

$$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} - (-3) \cdot 4x^{-4}$$

$$\frac{dy}{dx} = \frac{2}{3\sqrt[3]{x}} + \frac{12}{x^4}$$

$$15. y = 4(\sqrt{x} + 1)^5$$

Use product rule where; $\frac{d}{dx}(uv) = u dv + v du$ and chain rule where; $\frac{d}{dx}(u^n) = n du(u)^{n-1}$

$$u = 4$$

$$du = 0$$

$$v = \sqrt{x} + 1$$

$$dv = \frac{1}{2\sqrt{x}} dx$$

$$\frac{dy}{dx} = (4)(5) \left(\frac{1}{2\sqrt{x}}\right) (\sqrt{x} + 1)^4 + (0)(\sqrt{x} + 1)^5$$

$$\frac{dy}{dx} = \frac{10(\sqrt{x} + 1)^4}{\sqrt{x}}$$

$$16. y = \frac{4}{\sqrt{5x+3}}$$

Use quotient rule where; $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vdu - u dv}{v^2}$, and power rule where; $\frac{d}{dx}(\sqrt{u}) = \frac{du}{2\sqrt{u}}$

$$u = 4$$

$$du = 0$$

$$v = \sqrt{5x+3}$$

$$dv = \frac{5}{2\sqrt{5x+3}} dx$$

$$\frac{dy}{dx} = \frac{(0)\sqrt{5x+3} - \frac{10}{\sqrt{5x+3}}}{(\sqrt{5x+3})^2}$$

$$\frac{dy}{dx} = \frac{-\frac{10}{\sqrt{5x+3}}}{5x+3}$$

$$\frac{dy}{dx} = -\frac{10}{\sqrt{(5x+3)^3}}$$

$$17. y = \frac{2}{(4x+1)^3}$$

Use product rule where; $\frac{d}{dx}(uv) = u dv + v du$, and chain rule where; $\frac{d}{dx}(u^n) = n du(u)^{n-1}$

$$u = 2$$

$$du = 0$$

$$v = (4x+1)^{-3}$$

$$dv = -12(4x+1)^{-4} dx$$

$$\frac{dy}{dx} = -24(4x+1)^{-4} + (0)(4x+1)^{-3}$$

$$\frac{dy}{dx} = -24(4x+1)^{-4}$$

$$\frac{dy}{dx} = -\frac{24}{(4x+1)^4}$$

Evaluate $\frac{dy}{dx}$ at the specified value of x.

$$18. y = 6(\sqrt[3]{x} + 2)^2, x = 8$$

$$u = 6$$

$$du = 0$$

$$v = (\sqrt[3]{x} + 2)^2$$

$$dv = \frac{2(\sqrt[3]{x} + 2)}{3\sqrt[3]{x^2}} dx$$

$$\frac{dy}{dx} = 6\left(\frac{2(\sqrt[3]{x} + 2)}{3\sqrt[3]{x^2}}\right) + (0)(\sqrt[3]{x} + 2)^2$$

$$\frac{dy}{dx} = \frac{4(\sqrt[3]{x} + 2)}{\sqrt[3]{x^2}}, x = 8$$

$$f'(8) = \frac{4(\sqrt[3]{8} + 2)}{\sqrt[3]{8^2}}$$

$$f'(8) = \frac{4(\sqrt[3]{8} + 2)}{\sqrt[3]{64}}$$

$$f'(8) = \frac{4(2+2)}{4}$$

$$f'(8) = 4$$

$$19. y = \sqrt{6 - \sqrt{x}}, x = 4$$

$$u = 6 - \sqrt{x}$$

$$du = -\frac{1}{2\sqrt{x}} dx$$

$$\frac{dy}{dx} = \left(-\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{2\sqrt{6-\sqrt{x}}}\right)$$

$$\frac{dy}{dx} = -\frac{1}{4\sqrt{x}\sqrt{6-\sqrt{x}}}$$

$$f'(4) = -\frac{1}{4\sqrt{4}\sqrt{6-\sqrt{4}}}$$

$$f'(4) = -\frac{1}{4(2)\sqrt{6-2}}$$

$$f'(4) = -\frac{1}{8\sqrt{4}}$$

$$f'(4) = -\frac{1}{16}$$

$$20. y = x^3 + 4x^{-1}, x = 1$$

$$\frac{dy}{dx} = 3x^2 - \frac{4}{x^2}, x = 1$$

$$f'(1) = 3(1)^2 - \frac{4}{1^2}$$

$$f'(1) = 3 - 4$$

$$f'(1) = -1$$

$$21. y = (2x-1)^3 + \frac{4}{\sqrt{3x-2}}, x=2$$

$$u = (2x-1)^3$$

$$du = 6(2x-1)^2 dx$$

$$v = 4(3x-2)^{-\frac{1}{2}}$$

$$dv = -6(3x-2)^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = 6(2x-1)^2 - \frac{6}{\sqrt{(3x-2)^3}}$$

$$f'(2) = 6(2(2)-1)^2 - \frac{6}{\sqrt{(3(2)-2)^3}}$$

$$f'(2) = 6(4-1)^2 - \frac{6}{\sqrt{(6-2)^3}}$$

$$f'(2) = 6(3)^2 - \frac{6}{\sqrt{4^3}}$$

$$f'(2) = 6(9) - \frac{6}{\sqrt{64}}$$

$$f'(2) = 54 - \frac{3}{4}$$

$$f'(2) = \frac{213}{4}$$

Find the slope of the tangent to the curve at the given point.

$$\frac{dy}{dx} = m \text{ or slope}$$

$$22. y = 7 - x^2 + 4x^3, (-1, 2)$$

$$\frac{dy}{dx} = -2x + 12x^2, x = -1$$

$$f'(-1) = -2(-1) + 12(-1)^2$$

$$f'(-1) = 2 + 12$$

$$f'(-1) = 14$$

$$23. y = x + 2x^{-1}, (2, 3)$$

$$\frac{dy}{dx} = 1 - \frac{2}{x^2}, x = 2$$

$$f'(2) = 1 - \frac{2}{(2)^2}$$

$$f'(2) = 1 - \frac{2}{4}$$

$$f'(2) = 1 - \frac{1}{2}$$

$$f'(2) = \frac{1}{2}$$

$$24. y = 3x^2 - \frac{4}{x}, (2, 10)$$

$$\frac{dy}{dx} = 6x + \frac{4}{x^2}, x = 2$$

$$f'(2) = 6(2) + \frac{4}{(2)^2}$$

$$f'(2) = 12 + \frac{4}{4}$$

$$f'(2) = 12 + 1$$

$$f'(2) = 13$$

$$25. y = \frac{\sqrt{10-2x}}{3x}, \left(3, \frac{2}{9}\right)$$

$$u = \sqrt{10-2x}$$

$$du = -\frac{1}{\sqrt{10-2x}} dx$$

$$v = 3x$$

$$dv = 3 dx$$

$$\frac{dy}{dx} = \frac{3x \left(-\frac{1}{\sqrt{10-2x}} \right) - 3\sqrt{10-2x}}{9x^2}$$

$$\frac{dy}{dx} = -\frac{3(10-2x)+3x}{9x^2}$$

$$\frac{dy}{dx} = -\frac{30-6x+3x}{9x^2\sqrt{10-2x}}$$

$$\frac{dy}{dx} = -\frac{30-3x}{9x^2\sqrt{10-2x}}, x=3$$

$$f'(3) = -\frac{30-3(3)}{9(3)^2\sqrt{10-2(3)}}$$

$$f'(3) = -\frac{30-9}{9(9)\sqrt{10-6}}$$

$$f'(3) = -\frac{21}{81\sqrt{4}}$$

$$f'(3) = -\frac{21}{81(2)}$$

$$f'(3) = -\frac{21}{162}$$

$$f'(3) = -\frac{7}{54}$$

Find the values of x for which the derivative is zero.

$$26. y = x^3 + 4x^2 - 3x - 5$$

$$\frac{dy}{dx} = 3x^2 + 8x - 3, \quad \frac{dy}{dx} = 0$$

$$0 = 3x^2 + 8x - 3$$

Use quadratic equation to solve for x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a = 3, b = 8, c = -3$$

$$= \frac{-8 \pm \sqrt{8^2 - 4(3)(-3)}}{2(3)}$$

$$= \frac{-8 \pm \sqrt{64 + 36}}{6}$$

$$= \frac{-8 \pm \sqrt{100}}{6}$$

$$x = \frac{-8 + 10}{6} \approx \frac{1}{3}$$

$$x = \frac{-8 - 18}{6} \approx -3$$

$$27. y = x^4 - 8x^3 + 22x^2 - 24x + 9$$

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24, \quad \frac{dy}{dx} = 0$$

$$0 = x^3 - 6x^2 + 11x - 6$$

GCF of -6 is -1, 1, 6, -6, 2, -2, 3, -3

To check, it must be equal to zero.

$$x^3 - 6x^2 + 11x - 6 = 0$$

$$(1)^3 - 6(1)^2 + 11(1) - 6 = 0$$

$$1 - 6 + 11 - 6 = 0$$

$$-12 + 12 = 0$$

$$0 = 0$$

$$x = 1$$

$$(x-1)(x^2 - 5x + 6) = 0$$

$$(x-1)(x-2)(x-3) = 0$$

$$x = 1$$

$$x = 2$$

$$x = 3$$

$$28. y'' = 12x + 8x^{-1}$$

$$\frac{d^3y}{dx^3} = 12 - \frac{8}{x^2}$$

$$12 = \frac{8}{x^2}$$

$$12x^2 = 8$$

$$x^2 = \frac{8}{12}$$

$$x = \sqrt{\frac{2}{3}}$$

$$x = \pm \frac{\sqrt{6}}{3}$$

$$29. y = \frac{x-1}{x^2-2x+5}$$

$$u = x-1$$

$$du = dx$$

$$v = x^2 - 2x + 5$$

$$dv = (2x-2)dx$$

$$\frac{dy}{dx} = \frac{(x^2-2x+5) - (x-1)(2x-2)}{(x^2-2x+5)^2}$$

$$\frac{dy}{dx} = \frac{x^2-2x+5-2x^2+4x-2}{(x^2-2x+5)^2}$$

$$\frac{dy}{dx} = \frac{-x^2+2x+3}{(x^2-2x+5)^2}$$

$$0 = -\frac{x^2-2x-3}{(x^2-2x+5)^2}$$

$$-\frac{(x-3)(x+1)}{(x^2-2x+5)^2} = 0$$

$$x = 3$$

$$x = -1$$

Find the values of x given that;

$$30. y = 2x - 3x^{-1} \text{ and } \frac{dy}{dx} = 14$$

$$\frac{dy}{dx} = 2 + \frac{3}{x^2}$$

$$14 = 2 + \frac{3}{x^2}$$

$$12 = \frac{3}{x^2}$$

$$12x^2 = 3$$

$$x^2 = \frac{1}{4}$$

$$x = \pm \frac{1}{2}$$

$$31. y = x^{\frac{2}{3}} - x^{\frac{1}{3}} \text{ and } \frac{dy}{dx} = \frac{1}{4}$$

$$y = \sqrt[3]{x^2} - \sqrt[3]{x}$$

$$u = \sqrt[3]{x^2}$$

$$du = \frac{2}{3\sqrt[3]{x}} dx$$

$$v = x^{\frac{1}{3}}$$

$$dv = \frac{1}{3\sqrt[3]{x^2}} dx$$

$$\frac{dy}{dx} = \frac{2}{3\sqrt[3]{x}} - \frac{1}{3\sqrt[3]{x^2}}$$

$$\frac{1}{4} = \frac{2}{3\sqrt[3]{x}} - \frac{1}{3\sqrt[3]{x^2}}$$

$$\frac{1}{4} = \frac{2}{3x^{\frac{1}{3}}} - \frac{1}{3x^{\frac{1}{3} \cdot 2}}$$

Let $x^{\frac{1}{3}} = u$

$$\frac{1}{4} = \frac{2}{3u} - \frac{1}{3u^2}$$

$$\frac{1}{4} = \frac{6u^2 - 3u}{9u^3}$$

$$\frac{1}{4} = \frac{3u(2u-1)}{9u^3}$$

$$\frac{1}{4} = \frac{2u-1}{3u^2}$$

$$8u-4 = 3u^2$$

$$3u^2 - 8u + 4 = 0$$

$$3u^2 - 2u - 6u + 4 = 0$$

$$(3u-2)(u-2) = 0$$

$$u = \frac{2}{3}$$

Substitute the value of u

$$u = 2$$

$$x^{\frac{1}{3}} = \frac{2}{3}$$

Cube both side of the equation

$$x^{\frac{1}{3}} = 2$$

$$x = \frac{8}{27}$$

$$x = 8$$

$$32.y = 3x^2 + 4x^{-1} \text{ and } \frac{dy}{dx} = 11$$

$$\frac{dy}{dx} = 6x - \frac{4}{x^2}$$

$$11 = 6x - \frac{4}{x^2}$$

$$-\frac{4}{x^2} - 11 + 6x = 0$$

$$\frac{6x^3 - 11x^2 - 4}{x^2} = 0$$

$$(x-2)(6x^2+x+2) = 0$$

$$x = 2$$

EXERCISE 2.3

Use the Chain Rule to find $\frac{dy}{dx}$ and express the final answer in terms of x.

$$1. y = u^2 + u, \quad u = 2x + 1$$

$$y = (2x + 1)^2 + 2x + 1$$

$$\frac{dy}{dx} = 2(2)(2x + 1) + 2$$

$$\frac{dy}{dx} = 4(2x + 1) + 2$$

$$\frac{dy}{dx} = 8x + 6$$

$$2. y = \sqrt{u^2 - 1}, \quad u = 4\sqrt{x}$$

$$y = \sqrt{(4(\sqrt{x}))^2 - 1}$$

$$y = \sqrt{16x - 1}$$

$$\frac{dy}{dx} = \frac{16}{2\sqrt{16x-1}}$$

$$\frac{dy}{dx} = \frac{8}{\sqrt{16x-1}}$$

$$3. y = (u-4)^{\frac{3}{2}}, u = x^2 + 4$$

$$y = ((x^2 + 4) - 4)^{\frac{3}{2}}$$

$$y = (x^2)^{\frac{3}{2}}$$

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

$$4. y = (2u-2)^{\frac{2}{3}}, u = 4x^3 + 1$$

$$y = (2(4x^3 + 1) - 2)^{\frac{2}{3}}$$

$$y = (8x^3)^{\frac{2}{3}}$$

$$y = 4x^2$$

$$\frac{dy}{dx} = 8x$$

$$5. y = \sqrt{u+2}, u = 4x-2$$

$$y = \sqrt{4x-2+2}$$

$$y = 2\sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

$$6. y = \frac{2u}{u^2-1}, u = x^2$$

$$y = \frac{2x^2}{x^4-1}$$

$$v = 2x^2$$

$$dv = 4x dx$$

$$w = x^4 - 1$$

$$dw = 4x^3 dx$$

$$\frac{dy}{dx} = \frac{4x(x^4-1) - 2x^2(4x^3)}{(x^4-1)^2}$$

$$\frac{dy}{dx} = \frac{4x^5 - 4x - 8x^5}{(x^4-1)^2}$$

$$\frac{dy}{dx} = -\frac{4x(x^4+1)}{(x^4-1)^2}$$

$$7. y = \sqrt{u}, u = \sqrt{x}$$

$$y = \sqrt{\sqrt{x}}$$

$$y = \sqrt[4]{x}$$

$$\frac{dy}{dx} = \frac{1}{4\sqrt[4]{x^3}}$$

Use the Inverse Function Rule to find $\frac{dy}{dx}$

$$8. x = y + y^2 + y^3$$

$$\frac{dx}{dy} = 1 + 2y + 3y^2$$

$$\frac{dy}{dx} = \frac{1}{1 + 2y + 3y^2}$$

$$9. x = \sqrt{y} + \sqrt[3]{y}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}} + \frac{1}{3\sqrt[3]{y^2}}$$

$$\frac{dx}{dy} = \frac{3\sqrt[3]{y^2} + 2\sqrt{y}}{6\sqrt[6]{y^7}}$$

$$\frac{dy}{dx} = \frac{6\sqrt[6]{y^7}}{3\sqrt[3]{y^2} + 2\sqrt{y}}$$

$$10. x = (4 - 3y)^{\frac{3}{2}}$$

$$\frac{dx}{dy} = -\frac{9\sqrt{4-3y}}{2}$$

$$\frac{dy}{dx} = -\frac{2}{9\sqrt{4-3y}}$$

$$11. x = 2(4y+1)^3$$

$$\frac{dx}{dy} = 24(4y+1)^2$$

$$\frac{dy}{dx} = \frac{1}{24(4y+1)^2}$$

$$12. x = \frac{6}{(3y+1)^2}$$

$$u = 6$$

$$du = 0$$

$$v = (3y+1)^2$$

$$dv = 6(3y+1)dy$$

$$\frac{dx}{dy} = \frac{(0)(3y+1)^2 - 6(6(3y+1))}{(3y+1)^4}$$

$$\frac{dx}{dy} = -\frac{36(3y+1)}{(3y+1)^4}$$

$$\frac{dx}{dy} = -\frac{36}{(3y+1)^3}$$

$$\frac{dy}{dx} = -\frac{(3y+1)^3}{36}$$

$$13. x = \sqrt{1 + \sqrt{1 + \sqrt{y}}}$$

$$u = \sqrt{1 + \sqrt{y}}$$

$$du = \frac{1}{4\sqrt{1 + \sqrt{y}} \cdot \sqrt{y}} dy$$

$$\frac{dx}{dy} = \frac{1}{8\sqrt{1 + \sqrt{1 + \sqrt{y}}} \cdot \sqrt{1 + \sqrt{y}} \cdot \sqrt{y}}$$

$$\frac{dy}{dx} = 8\sqrt{1 + \sqrt{1 + \sqrt{y}}} \cdot \sqrt{1 + \sqrt{y}} \cdot \sqrt{y}$$

$$14. x = \left(\frac{2y+1}{3y-1}\right)^4$$

$$u = 2y+1$$

$$du = 2dy$$

$$v = 3y-1$$

$$dv = 3$$

$$\frac{dx}{dy} = 4 \left(\frac{2(3y-1) - 3(2y+1)}{(3y-1)^2} \right) \left(\frac{2y+1}{3y-1} \right)^3$$

$$\frac{dx}{dy} = 4 \left(\frac{6y-2-6y-3}{(3y-1)^2} \right) \left(\frac{2y+1}{3y-1} \right)^3$$

$$\frac{dx}{dy} = \left(-\frac{20}{(3y-1)^2} \right) \left(\frac{2y+1}{3y-1} \right)^3$$

$$\frac{dx}{dy} = -\frac{20(2y+1)^3}{(3y-1)^5}$$

$$\frac{dy}{dx} = -\frac{(3y-1)^5}{20(2y+1)^3}$$

EXERCISE 2.4

Find the first and second derivative of each of the following;

1. $y = x^5 + 3x^{-2} + 4x$

$$\frac{dy}{dx} = 5x^4 - \frac{6}{x^3} + 4$$

$$\frac{d^2y}{dx^2} = 20x^3 + \frac{18}{x^4}$$

2. $y = \frac{1}{x}$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

3. $y = \sqrt{4-x^2}$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{4-x^2}}$$

$$u = x$$

$$du = dx$$

$$v = \sqrt{4-x^2}$$

$$dv = -\frac{x}{\sqrt{4-x^2}} dx$$

$$\frac{d^2y}{dx^2} = -\frac{\frac{x}{\sqrt{4-x^2}} + \frac{x^2}{\sqrt{4-x^2}}}{4-x^2+x^2}$$

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{4-x^2}}{4-x^2}$$

$$\frac{d^2y}{dx^2} = -\frac{4}{\sqrt{(4-x^2)^2}}$$

4. $y = \frac{4x}{x+1}$

$$u = 4x$$

$$du = 4 dx$$

$$v = x+1$$

$$dv = dx$$

$$\frac{dy}{dx} = \frac{4(x+1) - 4x}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{(x+1)^2}$$

$$\frac{d^2y}{dx^2} = -\frac{2}{(x+1)^3}$$

5. $y = (x+5)^2$

$$\frac{dy}{dx} = 2(x+5)$$

$$\frac{d^2y}{dx^2} = 2$$

6. $y = \left(a^{\frac{1}{2}} - x^{\frac{1}{2}}\right)^2$

$$\frac{dy}{dx} = -\frac{\sqrt{a} - \sqrt{x}}{\sqrt{x}}$$

$$u = \sqrt{a} - \sqrt{x}$$

$$du = -\frac{1}{2\sqrt{x}} dx$$

$$v = \sqrt{x}$$

$$dv = \frac{1}{2\sqrt{x}}$$

$$\frac{d^2y}{dx^2} = -\frac{-\frac{\sqrt{x}}{2\sqrt{x}} - \frac{\sqrt{a} - \sqrt{x}}{2\sqrt{x}}}{x}$$

$$\frac{d^2y}{dx^2} = \frac{\sqrt{a}}{2x\sqrt{x}}$$

$$7. y = \frac{1 + \sqrt{x}}{\sqrt{x}}$$

$$u = 1 + \sqrt{x}$$

$$du = \frac{1}{2\sqrt{x}} dx$$

$$v = \sqrt{x}$$

$$dv = \frac{1}{2\sqrt{x}} dx$$

$$\frac{dy}{dx} = \frac{\frac{\sqrt{x}}{2\sqrt{x}} - \frac{1 + \sqrt{x}}{2\sqrt{x}}}{x}$$

$$\frac{dy}{dx} = -\frac{1}{2x\sqrt{x}}$$

$$u = 1$$

$$du = 0$$

$$v = 2x\sqrt{x}$$

$$dv = 3\sqrt{x} dx$$

$$\frac{d^2y}{dx^2} = -\frac{(0) \cdot 2x\sqrt{x} - 3\sqrt{x}}{4x^3}$$

$$\frac{d^2y}{dx^2} = \frac{3\sqrt{x}}{4x^3}$$

$$\frac{d^2y}{dx^2} = \frac{3}{4x^2\sqrt{x}}$$

$$8. y = \frac{x}{\sqrt{x-1}}$$

$$u = x$$

$$du = dx$$

$$v = \sqrt{x-1}$$

$$dv = \frac{1}{2\sqrt{x-1}} dx$$

$$\frac{dy}{dx} = \frac{\sqrt{x-1} - \frac{x}{2\sqrt{x-1}}}{x-1}$$

$$\frac{dy}{dx} = \frac{2(x-1) - x}{2\sqrt{(x-1)^3}}$$

$$\frac{dy}{dx} = \frac{x-2}{2\sqrt{(x-1)^3}}$$

$$u = x-2$$

$$du = dx$$

$$v = 2\sqrt{(x-1)^3}$$

$$dv = 3\sqrt{x-1}$$

$$\frac{d^2y}{dx^2} = \frac{2\sqrt{(x-1)^3} - (x-2)3\sqrt{x-1}}{4(x-1)^3}$$

$$\frac{d^2y}{dx^2} = \frac{\sqrt{x-1}(4-x)}{4(x-1)^3}$$

$$\frac{d^2y}{dx^2} = \frac{4-x}{4\sqrt{(x-1)^5}}$$

$$9. y = \frac{x^2}{x+1}$$

$$u = x^2$$

$$du = 2x dx$$

$$v = x+1$$

$$dv = dx$$

$$\frac{dy}{dx} = \frac{2x(x+1) - x^2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{2x^2 + 2x - x^2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{x^2 + 2x}{(x+1)^2}$$

$$u = x^2 + 2x$$

$$du = (2x+2) dx$$

$$v = (x+1)^2$$

$$dv = 2(x+1) dx$$

$$\frac{d^2y}{dx^2} = \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2+2x)}{(x+1)^4}$$

$$\frac{d^2y}{dx^2} = \frac{(x+1)[(2x+2)(x+1) - 2(x^2+2x)]}{(x+1)^4}$$

$$\frac{d^2y}{dx^2} = \frac{[2x^2 + 4x + 2 - 2x^2 - 4x]}{(x+1)^3}$$

$$\frac{d^2y}{dx^2} = \frac{2}{(x+1)^3}$$

10. If $y = \sqrt[3]{x}$, find $f'(8)$ and $f''(8)$.

$$y = \sqrt[3]{x}$$

$$\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}$$

$$f'(8) = \frac{1}{3\sqrt[3]{8^2}}$$

$$= \frac{1}{3\sqrt[3]{64}}$$

$$f'(8) = \frac{1}{12}$$

$$\frac{d^2y}{dx^2} = -\frac{2}{9\sqrt[3]{x^5}}$$

$$\frac{d^2y}{dx^2} = -\frac{2}{9x\sqrt[3]{x^2}}$$

$$f''(8) = -\frac{2}{9(8)\sqrt[3]{8^2}}$$

$$= -\frac{2}{72\sqrt[3]{64}}$$

$$f''(8) = -\frac{1}{144}$$

11. If $y = x^5$, find y^4 and $y^{(4)}$.

$$y = x^5$$

$$y^4 = x^{20}$$

$$y = x^5$$

$$\frac{dy}{dx} = 5x^4$$

$$\frac{d^2y}{dx^2} = 20x^3$$

$$\frac{d^3y}{dx^3} = 60x^2$$

$$\frac{d^4y}{dx^4} = 120x$$

12. Find the point on the curve $y = x^3 + 3x$ for which $y' = y''$

$$y = x^3 + 3x$$

$$\frac{dy}{dx} = 3x^2 + 3$$

$$\frac{d^2y}{dx^2} = 6x$$

$$3x^2 + 3 = 6x$$

$$x^2 + 1 = 2x$$

$$x^2 - 2x + 1 = 0$$

$$(x-1)(x-1) = 0$$

$$x = 1$$

$$f'(1) = 3x^2 + 3$$

$$= 3(1)^2 + 3$$

$$= 3 + 3$$

$$f'(1) = 6$$

$$f''(1) = 6x$$

$$= 6(1)$$

$$f''(1) = 6$$

$$y' = y''$$

$$y = x^3 + 3x$$

$$= (1)^3 + 3(1)$$

$$= 4$$

$$(1, 4)$$

13. How fast does the slope of the curve $y = (x^2 + x + 1)^2$ change at the point where $x=2$?

$$y = (x^2 + x + 1)^2$$

$$\frac{dy}{dx} = 2(2x+1)(x^2+x+1)$$

$$\frac{dy}{dx} = (4x+2)(x^2+x+1)$$

$$\frac{dy}{dx} = 4x^3 + 4x^2 + 4x + 2x^2 + 2x + 2$$

$$\frac{dy}{dx} = 4x^3 + 6x^2 + 6x + 2$$

$$f'(2) = 4(2)^3 + 6(2)^2 + 6(2) + 2$$

$$f'(2) = 4(8) + 6(4) + 6(2) + 2$$

$$f'(2) = 32 + 24 + 12 + 2$$

$$f'(2) = 70$$

14. Find the rate of change of the slope of the curve $y = x^3 - 1$ at $(2, 7)$

$$y = x^3 - 1$$

$$\frac{dy}{dx} = 3x^2$$

$$f'(2) = 3(2)^2$$

$$f'(2) = 12$$

Find $\frac{dy}{dx}$ by implicit differentiation.

$$1. x^3 + y^3 - 6xy = 0$$

$$3x^2 + 3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} - 6y = 0$$

$$\frac{dy}{dx} (3y^2 - 6x) = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

$$2. x^2 + xy^2 + y^2 = 1$$

$$2x + 2xy \frac{dy}{dx} + y^2 + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} (2xy + 2y) = -y^2 - 2x$$

$$\frac{dy}{dx} = \frac{-y^2 - 2x}{2y(x+1)}$$

$$\frac{dy}{dx} = -\frac{y^2 + 2x}{2y(x+1)}$$

$$3. \sqrt{x+y} + xy = 21$$

$$\frac{1 + \frac{dy}{dx}}{2\sqrt{x+y}} + x \frac{dy}{dx} + y = 0$$

$$\frac{1 + \frac{dy}{dx} + 2x\sqrt{x+y} \frac{dy}{dx}}{2\sqrt{x+y}} = -y$$

$$\frac{1}{2\sqrt{x+y}} + \frac{dy}{dx} \left(\frac{2x\sqrt{x+y} + 1}{2\sqrt{x+y}} \right) = -y$$

$$\frac{dy}{dx} \left(\frac{2x\sqrt{x+y} + 1}{2\sqrt{x+y}} \right) = -\left(y + \frac{1}{2\sqrt{x+y}} \right)$$

$$\frac{dy}{dx} \left(\frac{2x\sqrt{x+y} + 1}{2\sqrt{x+y}} \right) = -\frac{2y\sqrt{x+y} + 1}{2\sqrt{x+y}}$$

$$\frac{dy}{dx} = -\frac{(2y\sqrt{x+y} + 1)(2\sqrt{x+y})}{(2\sqrt{x+y})(2x\sqrt{x+y} + 1)}$$

$$\frac{dy}{dx} = -\frac{2y\sqrt{x+y} + 1}{2x\sqrt{x+y} + 1}$$

$$4. \sqrt{x} + \sqrt{y} = \sqrt{a}$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{y}} \frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$5. b^2 x^2 + a^2 y^2 = a^2 b^2$$

$$2b^2 x + 2a^2 y \frac{dy}{dx} = 0$$

$$2a^2 y \frac{dy}{dx} = -2b^2 x$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

$$6. (x-y)^3 = (x+y)^2$$

$$3 \left(1 - \frac{dy}{dx} \right) (x-y)^2 = 2 \left(1 + \frac{dy}{dx} \right) (x+y)$$

$$3(x-y)^2 - 3(x-y)^2 \frac{dy}{dx} = 2(x+y) + 2(x+y) \frac{dy}{dx}$$

$$3(x-y)^2 \frac{dy}{dx} + 2(x+y) \frac{dy}{dx} = 3(x-y)^2 - 2(x+y)$$

$$\frac{dy}{dx} (3(x-y)^2 + 2(x+y)) = 3(x-y)^2 - 2(x+y)$$

$$\frac{dy}{dx} = \frac{3(x-y)^2 - 2(x+y)}{3(x-y)^2 + 2(x+y)}$$

$$7. y = 4(x^2 + y^2)$$

$$y = 4x^2 + 4y^2$$

$$\frac{dy}{dx} = 8x + 8y \frac{dy}{dx}$$

$$\frac{dy}{dx} - 8y \frac{dy}{dx} = 8x$$

$$\frac{dy}{dx} (1 - 8y) = 8x$$

$$\frac{dy}{dx} = \frac{8x}{1 - 8y}$$

$$8. y^2 = \frac{3x+1}{2x-3}$$

$$u = 3x+1$$

$$du = 3dx$$

$$v = 2x - 3$$

$$dv = 2 dx$$

$$2y \frac{dy}{dx} = \frac{3(2x-3) - 2(3x+1)}{(2x-3)^2}$$

$$\frac{dy}{dx} = \frac{6x-9-6x-2}{2y(2x-3)^2}$$

$$\frac{dy}{dx} = -\frac{11}{2y(2x-3)^2}$$

$$9. y^2 - 3x + 2y = 0$$

$$2y \frac{dy}{dx} - 3 + 2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} (2y+2) = 3$$

$$\frac{dy}{dx} = \frac{3}{2(y+1)}$$

Find y'' in each of the following.

$$11. xy = 32$$

$$x \frac{dy}{dx} + y = 0$$

$$x \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{d^2y}{dx^2} = -\frac{x \frac{dy}{dx} - y}{x^2}$$

$$\frac{d^2y}{dx^2} = -\frac{x \left(-\frac{y}{x}\right) - y}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2y}{x^2}$$

$$12. x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

$$\frac{2}{3\sqrt[3]{x}} + \frac{2}{3\sqrt[3]{y}} \cdot \frac{dy}{dx} = 0$$

$$\frac{2}{3\sqrt[3]{y}} \cdot \frac{dy}{dx} = -\frac{2}{3\sqrt[3]{x}}$$

$$\frac{dy}{dx} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}}$$

$$\frac{d^2y}{dx^2} = -\frac{\sqrt[3]{x} \left(\frac{1}{3\sqrt[3]{y^2}} \times -\frac{\sqrt[3]{y}}{\sqrt[3]{x}} \right) - \sqrt[3]{y} \left(\frac{1}{3\sqrt[3]{x^2}} \right)}{\sqrt[3]{x^2}}$$

$$\frac{d^2y}{dx^2} = -\frac{\frac{\sqrt[3]{y^2}}{3y} - \frac{\sqrt[3]{y}}{3\sqrt[3]{x^2}}}{\sqrt[3]{x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{3\sqrt[3]{x^2} \frac{\sqrt[3]{y^2}}{\sqrt[3]{y^2}} + 3y \frac{\sqrt[3]{y}}{\sqrt[3]{y}}}{9y \sqrt[3]{x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{\sqrt[3]{x^2 y^2} + y \sqrt[3]{y}}{3y \sqrt[3]{x^4}}$$

$$\frac{d^2y}{dx^2} = \frac{\sqrt[3]{y} (\sqrt[3]{x^2 y^2} + y)}{3y \sqrt[3]{x^4}}$$

$$\frac{d^2y}{dx^2} = \frac{\sqrt[3]{x^2 y^2} + y}{3\sqrt[3]{x^4 y^2}}$$

$$13. y^2 - 16x = 0$$

$$2y \frac{dy}{dx} - 16 = 0$$

$$2y \frac{dy}{dx} = 16$$

$$\frac{dy}{dx} = \frac{8}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{8 \frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{8\left(\frac{8}{y}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{64}{y^3}$$

$$14. x^2 - 2xy + 3y^2 = 4$$

$$2x - 2x \frac{dy}{dx} - 2y + 6y \frac{dy}{dx} = 0$$

$$-2x \frac{dy}{dx} + 6y \frac{dy}{dx} = 2y - 2x$$

$$\frac{dy}{dx} (6y - 2x) = 2y - 2x$$

$$\frac{dy}{dx} = \frac{y-x}{3y-x}$$

$$\frac{d^2y}{dx^2} = \frac{3(y-x)\left(\frac{dy}{dx} - 1\right) - (y-x)\left(3\frac{dy}{dx} - 1\right)}{(3y-x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{3\left(y\frac{dy}{dx} - y - x\frac{dy}{dx} + x\right) - \left(3y\frac{dy}{dx} - y - 3x\frac{dy}{dx} + x\right)}{(3y-x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{3\left(y\left(\frac{y-x}{3y-x}\right) - y - x\left(\frac{y-x}{3y-x}\right) + x\right) - \left(3y\left(\frac{y-x}{3y-x}\right) - y - 3x\left(\frac{y-x}{3y-x}\right) + x\right)}{(3y-x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{3y(y-x)}{3y-x} - 3y - \frac{3x(y-x)}{3y-x} + 3x - \frac{3y(y-x)}{3y-x} + y + \frac{3x(y-x)}{3y-x} - x}{(3y-x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2x - 2y}{(3y-x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2(x-y)}{(3y-x)^2}$$

$$15. 4x^2(3y^2) = 36$$

$$8x + 18y \frac{dy}{dx} = 0$$

$$18y \frac{dy}{dx} = -8x$$

$$\frac{dy}{dx} = -\frac{4x}{9y}$$

$$u = 4x$$

$$du = 4dx$$

$$v = 9y$$

$$dv = 9 \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -\frac{36y - 36x \frac{dy}{dx}}{81y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{36y - 36x\left(-\frac{4x}{9y}\right)}{81y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{36y + \frac{144x^2}{9y}}{81y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{144x^2 + 324y^2}{729y^3}$$

$$\frac{d^2y}{dx^2} = -\frac{9(16x^2 + 36y^2)}{9(81y^3)}$$

$$\frac{d^2y}{dx^2} = -\frac{16x^2 + 36y^2}{81y^3}$$

Find the slope of the curve at the given point.

$$16. 2x^3 + 2y^3 = 9xy \text{ at } (2, 1)$$

$$6x^2 + 6y^2 \frac{dy}{dx} = 9x \frac{dy}{dx} + 9y$$

$$6y^2 \frac{dy}{dx} - 9x \frac{dy}{dx} = 9y - 6x^2$$

$$\frac{dy}{dx} (6y^2 - 9x) = 9y - 6x^2$$

$$\frac{dy}{dx} = \frac{9y - 6x^2}{6y^2 - 9x}$$

$$\frac{dy}{dx} = \frac{3y - 2x^2}{2y^2 - 3x}$$

$$f'(2, 1) = \frac{3(1) - 2(2)^2}{3(1)^2 - (2)}$$

$$f'(2, 1) = \frac{3 - 8}{3 - 2}$$

$$17. 3x^2 - 5y^2 = 51 \text{ at } (3, 2)$$

$$3y^2 \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{3y^2}$$

$$f'(3, 2) = \frac{2(3)}{3(2)^2}$$

$$f'(3, 2) = \frac{6}{12}$$

$$f'(3, 2) = \frac{1}{2}$$

$$18. x^2 + 4\sqrt{xy} + y^2 = 25 \text{ at } (4, 1)$$

$$2x + \frac{2\left(x \frac{dy}{dx} + y\right)}{\sqrt{xy}} + 2y \frac{dy}{dx} = 0$$

$$\frac{2x \frac{dy}{dx}}{\sqrt{xy}} + \frac{2y}{\sqrt{xy}} + 2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} \left(\frac{2x}{\sqrt{xy}} + 2y \right) = -2x - \frac{2y}{\sqrt{xy}}$$

$$\frac{dy}{dx} \left(\frac{2x + 2y\sqrt{xy}}{\sqrt{xy}} \right) = -\frac{2x\sqrt{xy} + 2y}{\sqrt{xy}}$$

$$\frac{dy}{dx} = -\frac{(x\sqrt{xy} + y)(2\sqrt{xy})}{(y\sqrt{xy} + x)(2\sqrt{xy})}$$

$$\frac{dy}{dx} = -\frac{x\sqrt{xy} + y}{y\sqrt{xy} + x}$$

$$f'(4, 1) = -\frac{(4)\sqrt{(4)(1)} + 1}{(1)\sqrt{(4)(1)} + 4}$$

$$f'(4, 1) = -\frac{4\sqrt{4} + 1}{\sqrt{4} + 4}$$

$$f'(4, 1) = -\frac{9}{6}$$

$$f'(4, 1) = -\frac{3}{2}$$

$$19. x^3 + x^2y + y^3 = 9 \text{ at } (-1, 2)$$

$$3x^2 + x^2 \frac{dy}{dx} + 2xy + 3y^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} (x^2 + 3y^2) = -(3x^2 + 2xy)$$

$$\frac{dy}{dx} = -\frac{3x^2 + 2xy}{x^2 + 3y^2}$$

$$f'(-1, 2) = -\frac{3(-1)^2 + 2(-1)(2)}{(-1)^2 + 3(2)^2}$$

$$f'(-1, 2) = -\frac{3 - 4}{1 + 12}$$

$$f'(-1, 2) = \frac{1}{13}$$

$$20. \sqrt{3x} + \sqrt[3]{4y} = 5 \text{ at } (3, 2)$$

$$\frac{\sqrt{3}}{2\sqrt{x}} + \frac{\sqrt[3]{4}}{3\sqrt[3]{y^2}} \cdot \frac{dy}{dx} = 0$$

$$\frac{\sqrt[3]{4}}{3\sqrt[3]{y^2}} \cdot \frac{dy}{dx} = -\frac{\sqrt{3}}{2\sqrt{3x}}$$

$$\frac{dy}{dx} = -\frac{\sqrt{27}\sqrt[3]{y^2}}{2\sqrt{x}\sqrt[3]{4}}$$

$$f'(3, 2) = -\frac{\sqrt{27}\sqrt[3]{2^2}}{2\sqrt{3}\sqrt[3]{4}}$$

$$f'(3, 2) = -\frac{\sqrt{27}\sqrt[3]{4}}{2\sqrt{3}\sqrt[3]{4}}$$

$$f'(3, 2) = -\frac{\sqrt{27}}{\sqrt{12}}$$

$$f'(3, 2) = -\frac{3}{2}$$

21. A circle is drawn with its center at (8,0) and with radius r such that the circle cuts the ellipse $x^2 + 4y^2 = 16$ at right angles.

Find the radius of the circle.

$$\text{Equation of the circle } (x-8)^2 + y^2 = r^2$$

Make the Ellipse Equation into standard form: $x^2 + 4y^2 = 16$

$$\text{Make the Ellipse Equation equal to 1; } \frac{x^2}{16} + \frac{4y^2}{16} = \frac{16}{16}$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

Differentiate both equation

$$(x-8)^2 + y^2 = r^2$$

$$2(x-8) + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2(x-8)$$

$$\frac{dy}{dx} = -\frac{x-8}{y}$$

$$y' = -\frac{x-8}{y}$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\frac{x}{8} + \frac{y}{2} \frac{dy}{dx} = 0$$

$$\frac{y}{2} \frac{dy}{dx} = -\frac{x}{8}$$

$$\frac{dy}{dx} = -\frac{x}{4y}$$

$$y' = -\frac{x}{4y}$$

$$\frac{x-8}{y} \cdot \frac{x}{4y} = -1,$$

$$\frac{x(x-8)}{4y^2} = -1$$

$$x(x-8) = -4y^2$$

Find y^2 using the ellipse equation

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\frac{y^2}{4} = 1 - \frac{x^2}{16}$$

$$y^2 = 4 - \frac{x^2}{4}$$

$$y^2 = 4 - \frac{x^2}{4}$$

Substitute y^2

$$x(x-8) = -4\left(4 - \frac{x^2}{4}\right)$$

$$x(x-8) = -16 + x^2$$

$$-x^2 + x^2 - 8x = -16$$

$$x = 2$$

$$y^2 = 4 - \frac{x^2}{4}$$

$$y^2 = 4 - \frac{2^2}{4}$$

$$y^2 = 3$$

Finally, find the value of r in the circle equation

$$(x-8)^2 + y^2 = r^2$$

$$(2-8)^2 + 3 = r^2$$

$$(-6)^2 + 3 = r^2$$

$$36 + 3 = r^2$$

$$r^2 = 39$$

$$r = \sqrt{39}$$

22. The vertex of the parabola $y^2 = 9x$ is the center of an ellipse. The focus of the parabola is an end of the minor axis of the ellipse, and the parabola and ellipse intersect at the right angles. Find the equation of the ellipse.