

APPORTIONMENT AND VOTING SYSTEM

INTRODUCTION TO APPORTIONMENT

Apportionment measures the population so that seats in the U.S. House of Representatives can be correctly apportioned among the states. (<https://www.census.gov/history/www/reference/apportionment/>). It is a method of dividing the whole into various parts.

Apportionment has been initially introduced in the U.S. Constitution as early as 1790, when it required its Representatives to be *apportioned* among the several states according to their respective population. Since the first apportionment in the U.S. House of Representatives in 1790, there are various methods used in determining how many voters would be represented by each member of the House. The two most common methods that were widely used are Hamilton and Jefferson plans.

THE HAMILTON METHOD

This is the method introduced by Alexander Hamilton which uses *standard divisor, quota* and considers the largest decimal remainder.

Standard Divisor is derived from dividing the total number of population by the number of people to apportion.

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Number of people to apportion}}$$

Standard Quota is the whole number part of the quotient of a population divided by the standard divisor.

Example No. 1

Suppose a fictitious country, Escia which consists of thirteen states with a total population of 42,000, needs to apportion 45 representatives. The population of each state is presented in table at the right. Find how many members will represent each state.

Steps:

1. Find the standard divisor.

$$\text{Standard Divisor} = \frac{42,000}{45} = 933.33$$

No.	State	Population
1	Ascovoli	3,800
2	Beninca	2,500
3	Costeri	3,150
4	Espigo	2,100
5	Fuenlativa	9,150
6	Guidolvía	2,000
7	Larim	2,750
8	Modilia	4,050
9	Olimento	2,400
10	Santiavelas	3,100
11	Sevinea	2,700
12	Trevine	2,600
13	Versine	1,700
	Total	42,000

2. Divide the population of each state by the standard divisor and **round the quotient down** to a whole number to get the standard quota.

No.	State	Population	Standard Quota	Apportioned No. of Representatives
1	Ascovoli	3,800	4.07	4
2	Beninca	2,500	2.68	2
3	Costeri	3,150	3.38	3
4	Espigo	2,100	2.25	2
5	Fuenlativa	9,150	9.80	9
6	Guidolvia	2,000	2.14	2
7	Larim	2,750	2.95	2
8	Modilia	4,050	4.34	4
9	Olimento	2,400	2.57	2
10	Santiavelas	3,100	3.32	3
11	Sevinea	2,700	2.89	2
12	Trevine	2,600	2.79	2
13	Versine	1,700	1.82	1
	Total	42,000	45	38

**Notice that the total number of representatives is only 38, not 45 as required.*

3. If the required total number of representatives is not attained, consider adding 1 representative for each state with the largest decimal remainder. In this case, 7 representatives are needed to complete 45, thus an additional of 1 representative will be added to the 7 states with largest decimal remainder.

No.	State	Population (P)	Standard Quota $\left(\frac{P}{SD}\right)$	Number of Representatives	Additional Representatives	Apportioned Number of Representatives
1	Ascovoli	3,800	4.07	4		4
2	Beninca	2,500	2.68	2	1	3
3	Costeri	3,150	3.38	3		3
4	Espigo	2,100	2.25	2		2
5	Fuenlativa	9,150	9.80	9	1	10
6	Guidolvia	2,000	2.14	2		2
7	Larim	2,750	2.95	2	1	3
8	Modilia	4,050	4.34	4		4
9	Olimento	2,400	2.57	2	1	3
10	Santiavelas	3,100	3.32	3		3
11	Sevinea	2,700	2.89	2	1	3
12	Trevine	2,600	2.79	2	1	3
13	Versine	1,700	1.82	1	1	2
	Total	42,000	45	38		45

Example No. 2

A total of 12 student-representatives will be apportioned among the five colleges of Bataan Peninsula State University Main Campus. Use the Hamilton method to determine how many members will represent each college given the population of each in the table at the right.

College	Population
CEA	4058
CT	2921
CICT	2750
CNM	1350
CBA	1010
Total	12089

Solution:

1. Solve for the standard divisor.

$$\text{Standard Divisor (SD)} = \frac{12089}{12} = \mathbf{1007.42}$$

2. Find the quotient of population divided by the standard divisor.
3. Determine the standard quota of each college – the whole number part of the quotient in (2).
4. If the required total number of representatives is not attained, add 1 member for each college with the highest decimal remainder. In this case, the attained total number of representatives from standard quota is only 10, so an additional of 1 member will be added to 2 colleges with the highest decimal remainder (CT – 0.90 and CICT – 0.73) to complete the required 12 representatives.

College	Population (P)	Standard Quota $\left(\frac{P}{SD}\right)$	Number of Representatives	Additional Representatives	Apportioned Number of Representatives
CEA	4058	4.03	4		4
CT	2921	2.90	2	1	3
CICT	2750	2.73	2	1	3
CNM	1350	1.34	1		1
CBA	1010	1.00	1		1
Total	12089		10		12

THE JEFFERSON METHOD

The Hamilton method is considered a fair method of apportionment until the deficit or surplus of the required number of representatives resulting to adding one representative for the state with the largest decimal remainder and continuously doing so until the required number of representative is attained.

To minimize this, Thomas Jefferson proposed a new method of apportionment. This method uses a **modified standard divisor (MSD)** which is determined by trial and error. It is usually lesser than the standard divisor to yield a higher quota.

In *Example No. 1*, the standard divisor is 933.33 and yield a total of 38 representatives which is 7 representatives short from the total required number. Using the Jefferson method, the standard divisor can be modified, i.e. using 900 as MSD.

Divide the population of each state by the chosen MSD; **the modified quota is the whole number part of the quotient.**

No.	State	Population (P)	Quota	Modified Quota MSD = 900
1	Ascovoli	3,800	4	4
2	Beninca	2,500	2	2
3	Costeri	3,150	3	3
4	Espigo	2,100	2	2
5	Fuenlativa	9,150	9	10
6	Guidolvia	2,000	2	2
7	Larim	2,750	2	3
8	Modilia	4,050	4	4
9	Olimento	2,400	2	2
10	Santiavelas	3,100	3	3
11	Sevinea	2,700	2	3
12	Trevine	2,600	2	2
13	Versine	1,700	1	1
	Total	42,000	38	41

Microsoft Excel can be utilized in easily finding a modified quota by using the INT function. Adjusting the MSD by trial and error, the total number representatives by state will also be adjusted as well as its total.

The screenshot shows an Excel spreadsheet with the following data and formulas:

No.	State	Population	Standard Quota	Modified Quota
1	Ascovoli	3,800	4	4
2	Beninca	2,500	2	3
3	Costeri	3,150	3	3
4	Espigo	2,100	2	2
5	Fuenlativa	9,150	9	11
6	Guidolvia	2,000	2	2
7	Larim	2,750	2	3
8	Modilia	4,050	4	4
9	Olimento	2,400	2	2
10	Santiavelas	3,100	3	3
11	Sevinea	2,700	2	3
12	Trevine	2,600	2	3
13	Versine	1,700	1	2
	Total	42,000	38	45
	Required Number		45	
	SD		933.33	
	MSD		820	

Formulas shown in callouts:

- Standard Divisor (SD):** = Total Population / Number of Representatives = G19/G20
- Modified Quota:** = whole number part of Population of the state / modified standard divisor = INT(G6/G22)
- Standard Quota (SQ):** = whole number part of Population of the state / standard divisor = INT(G6/G21)
- Total Representatives:** = SUM(I6:I18)

Notice that when 900 is used as MSD, the total modified quota becomes 41 which is still not the required total number of representatives, thus another MSD should be used.

Trying 820 as MSD, the table at the right will yield 45 as the total number of representatives.

No.	State	Population (P)	Quota	Modified Quota MSD = 820
1	Ascovoli	3,800	4	4
2	Beninca	2,500	2	3
3	Costeri	3,150	3	3
4	Espigo	2,100	2	2
5	Fuenlativa	9,150	9	11
6	Guidolvia	2,000	2	2
7	Larim	2,750	2	3
8	Modilia	4,050	4	4
9	Olimento	2,400	2	2
10	Santiavelas	3,100	3	3
11	Sevinea	2,700	2	3
12	Trevine	2,600	2	3
13	Versine	1,700	1	2
	Total	42,000	38	45

College	Population (P)	(3) Quota	Modified Quota MSD = 900
CEA	4058	4	4
CT	2921	2	3
CICT	2750	2	3
CNM	1350	1	1
CBA	1010	1	1
Total	12089	10	12

Referring to *Example No. 02*, in Hamilton Method, the standard divisor derived is 1007.42 and yield a total number of representatives of 10 instead of the required 12.

In *Jefferson Method* a modified standard divisor of 900 can be used and the total number of representatives will yield 12.

Example No. 03

Suppose 30 delegates to the *Bantay Dagat Council* will be selected according to population of the six towns of the first district of Bataan as shown in the table below. Use the Hamilton and Jefferson method to determine the number of delegates from each town.

Town	Population*	SQ	Hamilton			Jefferson
			Quota	Additional	Adjusted	
Dinalupihan	106,371	9.27	9		9	10
Hermosa	65,862	5.74	5	1	6	6
Orani	66,909	5.83	5	1	6	6
Samal	35,298	3.08	3		3	3
Abucay	39,880	3.48	3		3	3
Morong	29,901	2.61	2	1	3	2
Total	344,221	30	27		30	30
Required Number of Delegates	30	SD = 11,474.03				MSD = 10,500

*Source: www.psa.gov.ph

THE WEBSTER METHOD

This method was introduced by Daniel Webster as an alternative to Jefferson method where quota rule may be violated. Using the Webster method, a modified standard divisor may be less than, equal to or greater than the standard divisor. Unlike the Jefferson method, the Webster method apply the rules for rounding off the quotient.

From *Example No. 1*, using the modified standard divisor of 920 instead of the standard divisor of 933.33 and rounding off the quotient of population of the state by this modified standard divisor, the total number of representatives will yield 45.

No.	State	Population (P)	Standard Quota $\left(\frac{P}{SD}\right)$	Hamilton SD = 933.33 (+1 to those with higher decimal part)	Jefferson MSD=820 (round down)	Webster MSD=920 (round off)
1	Ascovoli	3,800	4.07	4	4	4
2	Beninca	2,500	2.68	3	3	3
3	Costeri	3,150	3.38	3	3	3
4	Espigo	2,100	2.25	2	2	2
5	Fuenlativa	9,150	9.80	10	11	10
6	Guidolvia	2,000	2.14	2	2	2
7	Larim	2,750	2.95	3	3	3
8	Modilia	4,050	4.34	4	4	4
9	Olimento	2,400	2.57	3	2	3
10	Santiavelas	3,100	3.32	3	3	3
11	Sevinea	2,700	2.89	3	3	3
12	Trevine	2,600	2.79	3	3	3
13	Versine	1,700	1.82	2	2	2
	Total	42,000	45	45	45	45

From *Example No. 02*, the standard divisor is 1007.42. Using Hamilton method, the total number of representative was 10 resulting to adding 2 representatives. Using the same standard divisor and applying the Webster method – *rounding off* the quotient of population by the standard divisor, the resulting total number of representative is also 12. in this case, there is no need of modifying the standard divisor.

College	Population (P)	Standard Quota $\left(\frac{P}{SD}\right)$	Quota	Webster's Method of Apportionment
CEA	4058	4.03	4	4
CT	2921	2.90	2	3
CICT	2750	2.73	2	3
CNM	1350	1.34	1	1
CBA	1010	1.00	1	1
Total	12,089	12	10	12
Standard Divisor	1007.42			

HUNTINGTON-HILL APPORTION METHOD

This method is based on the apportionment principle and is called as the method of equal proportions. The initial procedure is the same with the other apportionment method – finding the standard divisor first.

The next step involves *rounding up* and *rounding down* the quotient (upper and lower quota), then finding the geometric mean (\sqrt{UL}).

If the geometric mean is greater than the standard quota, *round down* the geometric mean to get the number of representative. On the other hand, if the geometric mean is less than or equal the standard quota, *round up* the geometric mean.

To illustrate, referring to *Example No. 01* with standard divisor of 933.33 (*Total Population / Number of Representative to be apportioned*), find the upper and lower quota based on the standard quota of each state and solve for each geometric mean. The *upper quota* is the standard quota rounded up to a whole number, while the *lower quota* is the standard quota rounded down to a whole number.

Next step is to compare the geometric mean with the standard quota to get the number of representatives to be apportioned per state. If the total number of representatives is not equal to the required number, the standard divisor should be modified.

No.	State	Population (P)	Standard Quota $\left(\frac{P}{SD}\right)$	Upper Quota	Lower Quota	Geometric Mean \sqrt{UL}	Apportioned Number of Representatives
1	Ascovoli	3,800	4.07	5	4	4.47	4
2	Beninca	2,500	2.68	3	2	2.45	3
3	Costeri	3,150	3.38	4	3	3.46	3
4	Espigo	2,100	2.25	3	2	2.45	2
5	Fuenlativa	9,150	9.80	10	9	9.49	10
6	Guidolvia	2,000	2.14	3	2	2.45	2
7	Larim	2,750	2.95	3	2	2.45	3
8	Modilia	4,050	4.34	5	4	4.47	4
9	Olimento	2,400	2.57	3	2	2.45	3
10	Santiavelas	3,100	3.32	4	3	3.46	3
11	Sevinea	2,700	2.89	3	2	2.45	3
12	Trevine	2,600	2.79	3	2	2.45	3
13	Versine	1,700	1.82	2	1	1.41	2
	Total	42,000	45	51	38	43.92	45

Applying the Huntington-Hill Method to *Example No. 2*,

College	Population (P)	Standard Quota $\left(\frac{P}{SD}\right)$	Upper Quota	Lower Quota	Geometric Mean \sqrt{UL}	Apportioned Number of Representatives
CEA	4058	4.03	5	4	4.47	4
CT	2921	2.90	3	2	2.45	3
CICT	2750	2.73	3	2	2.45	3
CNM	1350	1.34	2	1	1.41	1
CBA	1010	1.00	1	1	1.00	1
Total	12,089	12	14	10	11.79	12
Standard Divisor	1007.42					

In case of *Example No. 03*, the required total number of delegates is not attained using the standard divisor, thus, a modified divisor should be used.

Town	Population*	Standard Quota $\left(\frac{P}{SD}\right)$	Upper Quota	Lower Quota	Geometric Mean \sqrt{UL}	Apportioned Number of Delegates
Dinalupihan	106,371	9.27	10	9	9.49	9
Hermosa	65,862	5.74	6	5	5.48	6
Orani	66,909	5.83	6	5	5.48	6
Samal	35,298	3.08	4	3	3.46	3
Abucay	39,880	3.48	4	3	3.46	4
Morong	29,901	2.61	3	2	2.45	3
Total	344,221	30	33	27	29.82	31
Standard Divisor	11474.03					

Using 11,600 as the modified divisor, the required total number of delegates which is 30, is attained.

Town	Population*	Standard Quota $\left(\frac{P}{SD}\right)$	Upper Quota	Lower Quota	Geometric Mean \sqrt{UL}	Apportioned Number of Delegates
Dinalupihan	106,371	9.17	10	9	9.49	9
Hermosa	65,862	5.68	6	5	5.48	6
Orani	66,909	5.77	6	5	5.48	6
Samal	35,298	3.04	4	3	3.46	3
Abucay	39,880	3.44	4	3	3.46	3
Morong	29,901	2.58	3	2	2.45	3
Total	344,221	29.67	33	27	29.82	30
Modified Divisor	11474.03					

FAIRNESS IN APPORTIONMENT

As any of the methods of apportionment will affect the number of representatives and may result to unfairness, there are criteria that should be satisfied to make the apportionment fair.

1. Quota Rule

The number of representatives apportioned to a state is the standard quota or one more than the standard quota.

Notice that Hamilton method does not violate quota rule – the number of representatives apportioned to a state is equal to the standard quota or one more than the standard quota.

On the other hand, in Jefferson method, the quota rule may sometimes be violated, i.e. in *Example No. 01*, the state of Fuenlativa which has a population of 9,150 has a quota of 9, but using the Jefferson method

2. Average Constituency

The ratio of the population of a state over the number of representatives from that state rounded to the nearest whole number.

State	Population	Standard Quota	Hamilton	Jefferson
Fuenlativa	9,150	9.80	10	11

$$\text{Average Constituency} = \frac{\text{population of a state}}{\text{number of representative from the state}}$$

This only shows how well a certain state is represented.

Example: Referring to *Example No. 02*, are the colleges well represented?

To find the Average Constituency, divide the population of each college by the number of its representative;

College	Population	Number of Representatives	Average Constituency
CEA	4058	4	1015
CT	2921	3	974
CICT	2750	3	917
CNM	1350	1	1350
CBA	1010	1	1010

This shows that CEA and CBA are equally represented since they have almost equal average constituency.

3. Absolute Unfairness of an Apportionment

This is the absolute value of the difference of the average constituency of two states.
 $= |\text{Average Constituency of State A} - \text{Average Constituency of State B}|$

This is used when an additional representative will be added.

Example: Referring again to *Example No. 2*, find the absolute unfairness of apportionment when one representative will be added to CNM or CBA. In which college does the representative will be added?

Solution: Solve for average constituency when one representative will be added either to CNM or CBA, find their absolute unfairness of apportionment and compare.

College	Average Constituency when the representative will be added to CBA	Average Constituency when the representative will be added to CNM
CNM	$\frac{1350}{1} = 1350$	$\frac{1350}{2} = 675$
CBA	$\frac{1010}{2} = 505$	$\frac{1010}{1} = 1010$
Absolute Unfairness of Apportionment	$ 1350 - 505 = \mathbf{845}$	$ 675 - 1010 = \mathbf{335}$

Since the absolute unfairness of apportionment is lesser when the representative will be added to CNM, then is fair that the additional representative will be for CNM.

4. Relative Unfairness of an Apportionment

This is the quotient of the absolute unfairness of the apportionment and the average constituency of the state receiving the new representative.

$$= \frac{\text{Absolute unfairness of the apportionment}}{\text{Average constituency of the state receiving the new representative}}$$

From the previous example, the relative unfairness of apportionment for CNM when one representative is added will be

$$= \frac{335}{675} = 0.4963$$

Example No. 04. The table at the right shows the number of barangay health workers (BHW) assigned in two barangays. Find the relative unfairness of apportionment if another BHW will be added to one of the barangays.

Barangay	Population	No. of BHW
Makakalikasan	3018	5
Makatao	3816	7

Solution:

- Determine the average constituency

Barangay	Population	No. of BHW	Average Constituency when the BHW will be at Makakalikasan	Average Constituency when the BHW will be at Makatao
Makakalikasan	3018	5	$\frac{3018}{6} = 503$	$\frac{3018}{5} = 604$
Makatao	3816	7	$\frac{3816}{7} = 545$	$\frac{3816}{8} = 477$

- Solve for the absolute unfairness of apportionment

Barangay	Population	No. of BHW	Average Constituency when the BHW will be at Makakalikasan	Average Constituency when the BHW will be at Makatao
Makakalikasan	3018	5	$\frac{3018}{6} = 503$	$\frac{3018}{5} = 604$
Makatao	3816	7	$\frac{3816}{7} = 545$	$\frac{3816}{8} = 477$
Absolute Unfairness of Apportionment			$503 - 545 = 42$	$604 - 477 = 127$

At this point, we can say that the additional BHW will be assigned to Barangay Makakalikasan since it has lower absolute unfairness or apportionment, and its **relative unfairness of apportionment** is

$$= \frac{\text{Absolute unfairness of the apportionment}}{\text{Average constituency of the state receiving the new representative}} = \frac{42}{503} = 0.0835$$

APPORTIONMENT PRINCIPLE

This states that when adding a new representative to a state, the representative is assigned to a state which will give the smallest relative unfairness of apportionment.

THE HUNTINGTON-HILL APPORTIONMENT PRINCIPLE

This states that if there is a need to add one representation to one of the several states, this should be added to one which has the greatest Huntington-Hill number.

The *Huntington-Hill Number* (HH Number) is equal to

$$\text{HH Number} = \frac{(P_A)^2}{a(a+1)}$$

where

P_A = population of State A
 a = the current number of representatives from State A

Using *Example No. 04*, determine at which barangay that one BHW should be added using the Huntington-Hill apportionment principle.

Solution: Calculate the HH Number of each barangay:

Barangay	Population	No. of BHW
Makalikasan	3018	5
Makatao	3816	7

Barangay	Population	No. of BHW	HH Number = $\frac{(P_A)^2}{a(a+1)}$
Makalikasan	3018	5	$\frac{(3018)^2}{5(5+1)} \approx 303,611$
Makatao	3816	7	$\frac{(3816)^2}{7(7+1)} \approx 260,033$

Brgy. Makalikasan has the greater HH number and according to Huntington-Hill apportionment principle, the additional BHW should be assigned to Brgy. Makalikasan

EXERCISES:

1. Two hundred (200) trash bins will be distributed among five (5) barangays of Morong. This will be apportioned according to their population as shown in the table at the right:

<i>Barangay</i>	<i>Population</i>
Binaritan	5,966
Mabayo	4,771
Nagbalayong	6,207
Poblacion	5,511
Sabang	7,446
Total	29,901
<i>Source: www.psa.gov.ph</i>	

- a. Determine the standard divisor.
 - b. Determine the standard quota of each barangay.
 - c. Find the number of trash bins which will be allocated per barangay using (c.1) the Hamilton Method; (c.2) the Jefferson Method; (c.3) the Webster Method; and (c.4) the Huntington Hill Method
 - d. Was quota rule violated in any of the methods used? Explain why or why not.
2. The table below shows the population of students per program in the College of Information and Communications Technology. Using the Huntington-Hill Method, determine the number of computers to be apportioned per program if there are 350 computers to be distributed.

Program	Population
BSCS	340
BSMIS	550
BSEMC	385
BSIT	725
Total	2000

- a. What is the average constituency if the additional computer will be given to BSCS? to BSEMC?
 - b. Find the absolute unfairness of apportionment.
 - c. Find the relative unfairness of apportionment.
 - d. Which program should receive the additional unit? Explain why.
4. Referring to previous number, solve for the Huntington-Hill number and determine if the same decision will be made.

INTRODUCTION TO VOTING

One of the basic rights that people in a democratic country enjoy is their ability to choose their leaders through voting. This is a formal expression of preference for a candidate for office or for a proposed resolution of an issue which generally influences decision making.

Voting generally takes place in the context of a large-scale national or regional election, however, local and small-scale community elections can be just as critical to individual participation in government. (<http://hrlibrary.umn.edu/edumat/studyguides/votingrights.html>)

METHODS OF VOTING

PLURALITY METHOD

Each voter votes for one candidate, and the candidate with the most votes wins. The winning candidate does not have to have a majority of the votes (over 50% of the number of voters). In case of a tie, a special runoff election maybe held. If votes are ranked, the candidate with the greatest number of first-place votes is declared as the winner

Plurality with Elimination Method (without Rank)

In this method, the candidate who receives majority of the votes is declared as winner at once but if no one receives majority of votes, the candidate with the least number of votes is eliminated and a new election is held. This process continues until a candidate receives a majority of votes.

Plurality with Elimination Method (with rank)

The candidate with the least number of first-place votes is eliminated first; if two or more candidates have the same number of first-place vote, then they will be the remaining candidates as another election is held, adjusting the rankings of the candidates. This process is repeated until only two candidate are left and the candidate with the majority of votes wins.

Example No. 01. Fifty students were asked to rank their most used social media networks using 1 as their most used and 5 as their \least used network. The table below shows the rankings of their preferences. Which social media network is the most used network according to plurality method?

Social Media	Rankings			
Facebook	1	3	2	1
Instagram	2	1	3	4
Snapchat	4	2	4	3
Twitter	3	4	1	2
Number of Voters	18	16	12	4

Solution: To show which network is the students' most used network, identify the number of first-place votes of each network. Referring to the table, Facebook is ranked first in the first column and fourth column which corresponds to 18 and 4 voters; Instagram ranked first in the second column which corresponds to 16 voters; while Twitter got first place in the third column which represents 12 voters.

Social Media	First-Place Votes
Facebook	$18 + 4 = 22$
Instagram	16
Snapchat	0
Twitter	12

This means Facebook got the most number of first-place votes thus it is the most used social network among 50 students by plurality method.

Plurality method with elimination (with rank) can also be applied in this example since the total first-place votes for Facebook is not the majority of votes (50% of 50 is 25, it has only 22 votes).

By this method, Snapchat will be eliminated and ranking will be adjusted.

Social Media	Rankings				First-Place Votes
Facebook	1	2	2	1	$18 + 4 = 22$
Instagram	2	1	3	3	16
Twitter	3	3	1	2	12
Number of Voters	18	16	12	4	

Again, majority of votes is still not attained, thus elimination will be done again. This time, Twitter will be eliminated and rankings will be adjusted.

Social Media	Rankings				First-Place Votes
Facebook	1	2	1	1	$18 + 12 + 4 = 34$
Instagram	2	1	2	2	16
Number of Voters	18	16	12	4	

This shows that Facebook is the most used social media network by plurality with elimination method.

BORDA COUNT METHOD

In this method of voting, points are assigned for each candidate's rank, i.e. if there are n candidates or issues in an election, each voter ranks the candidates or issues by giving n points to the voter's first choice, $n-1$ points for the voter's second choice, and so on, with the voter's least favorite choice receiving 1 point. The candidate or issue that receives the most total points is the winner.

Example No. 02. One hundred movie goers were asked to rank the following Avengers movies according to their preferences. Determine which Avenger movie is the most preferred by using the Borda Count Method.

Movie	Ranking				
Iron Man 3	3	1	4	6	1
Thor: Ragnarok	4	6	6	5	4
Black Panther	1	4	2	1	3
Captain America: Civil War	5	3	3	2	2
Spiderman: Homecoming	6	5	5	4	5
The Avengers: Infinity War	2	2	1	3	6
Number of Voters	38	29	15	12	6

Solution: Points should be assigned to corresponding ranks; highest point is 6 since there are six movies which were ranked

Rank 1 = 6 points Rank 2 = 5 points Rank 3 = 4 points
 Rank 4 = 3 points Rank 5 = 2 points Rank 6 = 1 point

Movie	No. of Voters * Corresponding Points/Rank				
Iron Man 3	38*4	29*6	15*3	12*1	6*6
Thor: Ragnarok	38*3	29*1	15*1	12*2	6*3
Black Panther	38*6	29*3	15*5	12*6	6*4
Captain America: Civil War	38*2	29*4	15*4	12*5	6*5
Spiderman: Homecoming	38*1	29*2	15*2	12*3	6*2
The Avengers: Infinity War	38*5	29*5	15*6	12*4	6*1
Number of Voters	38	29	15	12	6

Multiply the number of voters by the corresponding points of their rank then sum up the points to get the total number of points. The movie with the highest number of points is the most preferred Avenger movie of the respondents.

Movies	Points					Total Points	Rank
Iron Man 3	152	174	45	12	36	419	3
Thor: Ragnarok	114	29	15	24	18	200	5
Black Panther	228	87	75	72	24	486	1
Captain America: Civil War	76	116	60	60	30	342	4
Spiderman: Homecoming	38	58	30	36	12	174	6
The Avengers: Infinity War	190	145	90	48	6	479	2

Using this Borda count method, it shows that Black Panther is the most preferred Avenger movie of the respondents

Using Microsoft Excel

Movie	Ranking	Rank	Points
Iron Man	3 1 4 6 1	1	6
Thor: Ragnarok	4 6 6 5 4	2	5
Black Panther	1 4 2 1 3	3	4
Captain America: Civil War	5 3 3 2 2	4	3
Spiderman: Homecoming	6 5 5 4 5	5	2
The Avengers: Infinity War	2 2 1 3 6	6	1
Number of Voters	38 29 15 12 6		

Movies	Corresponding Points	Total Points	Rank
Iron Man	152 174 45 12 36	419	3
Thor: Ragnarok	114 29 15 24 18	200	5
Black Panther	228 87 75 72 24	486	1
Captain America: Civil War	76 116 60 60 30	342	4
Spiderman: Homecoming	38 58 30 36 12	174	6
The Avengers: Infinity War	190 145 90 48 6	479	2

range of points

Final ranking can be generated by using the RANK function.
`=RANK(H23,H23:H28,0)`

These columns can be automatically generated using multiplication and vlookup functions
`=number of voters * (VLOOKUP(rank, range of points, 2))`
`=C$20*(VLOOKUP(C14,$I$14:$I$19,2))`

This column is the sum of the corresponding points. These can be generated by using the SUM function.
`=SUM(C28:G28)`

PAIRWISE COMPARISON VOTING METHOD

The pairwise comparison method of voting is usually referred as the *head-to-head* method of voting. In this method, each candidate is compared one-to-one with each of the other candidates. A candidate receives 1 point for a win, 0.5 point for a tie and 0 point for a loss. A candidate with the highest points wins.

Example No. 03. Suppose a pairwise comparison method will be used to determine the winner of home owners association's presidency. Three hundred members of HOA were asked to rank the four candidates in order of their preferences. Determine who ranked first in the election using the table below:

Candidates	Rankings			
Mejia	4	1	2	4
Pantoja	2	3	1	2
Quintero	1	2	3	1
Soler	3	4	4	3
Number of Voters	108	85	65	42

Solution: By pairwise comparison method, each candidate will be paired with each of the other candidate and determine who was favored over the other.

Between Mejia and Pantoja:

Mejia	4	1	2	4	Total
Pantoja	2	3	1	2	
Number of Voters	108	85	65	42	
Mejia was favored over Pantoja		85			85
Pantoja was favored over Mejia	108		65	42	215

*Pantoja will get 1 point

Between Mejia and Quintero:

Mejia	4	1	2	4	Total
Quintero	1	2	3	1	
Number of Voters	108	85	65	42	
Mejia was favored over Quintero		85	65		150
Quintero was favored over Mejia	108			42	150

*Mejia and Quintero will get 0.5 point each since they get the same points.

Between Mejia and Pantoja:

Mejia	4	1	2	4	Total
Soler	3	4	4	3	
Number of Voters	108	85	65	42	
Mejia was favored over Soler		85	65		150
Soler was favored over Mejia	108			42	150

*Mejia and Soler will get 0.5 point each since they get the same points.

Between Pantoja and Quintero:

Pantoja	2	3	1	2	Total
Quintero	1	2	3	1	
Number of Voters	108	85	65	42	
Pantoja was favored over Quintero			65		65
Quintero was favored over Pantoja	108	85		42	235

*Quintero will get 1 point.

Between Pantoja and Soler:

Pantoja	2	3	1	2	Total
Soler	3	4	4	3	
Number of Voters	108	85	65	42	
Pantoja was favored over Soler	108	85	65	42	300
Soler was favored over Pantoja					0

*Pantoja will get 1 point.

Between Quintero and Soler:

Quintero	1	2	3	1	Total
Soler	3	4	4	3	
Number of Voters	108	85	65	42	
Quintero was favored over Soler	108	85	65	42	300
Soler was favored over Quintero					0

*Quintero will get 1 point.

To summarize their points:

Candidates	Number of Points
Mejia	$0.5 + 0.5 = 1$
Pantoja	$1 + 1 = 2$
Quintero	$0.5 + 1 + 1 = 2.5$
Soler	0.5

**Quintero wins the presidency.*

Example No. 04. A private organization is planning to treat thirty (30) senior citizens to a tour in one of the popular destinations in the Philippines. The officers of the organization asked the senior citizens to rank the (4) cities in order of their preferences.

Cities	Ranking			
Cebu	3	4	2	4
Davao	4	3	1	2
Ilocos	2	1	3	3
Palawan	1	2	4	1
Number of senior citizens	11	9	7	3

Determine which city is the most preferred destination of the senior citizens using the four methods of voting.

Solution:

- By using *plurality method*, **Palawan** is the most preferred city because it garnered 11 first-place votes.
- By using *plurality with elimination method*, Cebu will be eliminated first since it has no first-place votes. Adjusting the ranking of the remaining three cities:

Cities	Ranking				Number of First-Place votes
Davao	3	3	1	2	7
Ilocos	2	1	2	3	9
Palawan	1	2	3	1	$11 + 3 = 14$
Number of senior citizens	11	9	7	3	

The next city to be eliminated will be Davao since it has the lowest number of first-place votes.

Cities	Ranking				Number of First-Place votes
Ilocos	2	1	1	2	$9 + 7 = 16$
Palawan	1	2	2	1	$11 + 3 = 14$
Number of senior citizens	11	9	7	3	

At this point, Ilocos received 16 first-place votes while Palawan received 14 first-place votes. Therefore, by *plurality with elimination method*, **Ilocos** is the most preferred tourist destination among the senior citizens.

- c. By using *Borda count method*, corresponding points will be assigned to each of the four ranks: Rank 1 \approx 4 points; Rank 2 \approx 3 points; Rank 3 \approx 2 points; and Rank 4 \approx 1 point.

Cities	Ranking & corresponding points				Total Points	Final Rank
Cebu	3	4	2	4	55	4
	11*2 = 22	9*1 = 9	7*3 = 21	3*1 = 3		
Davao	4	3	1	2	66	3
	11*1 = 11	9*2 = 18	7*4 = 28	3*3 = 9		
Ilocos	2	1	3	3	89	2
	11*3 = 33	9*4 = 36	7*2 = 14	3*2 = 6		
Palawan	1	2	4	1	90	1
	11*4 = 44	9*3 = 27	7*1 = 7	3*4 = 12		
Number of senior citizens	11	9	7	3		

By using the *Borda count method*, Palawan got the highest number of points which is 90 points, thus, it is the most preferred tourist destination of the senior citizens.

- d. By using the *pairwise comparison method*, each city will be paired with each of the other city and determine who was favored over the other. In each pairing, the winner will get 1 point, 0.5 if they have equal votes.
- Between Cebu and Davao

Cities	Ranking					
Cebu	3	4	2	4	Cebu was favored over Davao	11

Davao	4	3	1	2	Davao was favored over Cebu	$9 + 7 + 3 = 19$
Number of senior citizens	11	9	7	3	<i>Davao will get 1 point</i>	

2. Between Cebu and Ilocos

Cities	Ranking					
Cebu	3	4	2	4	Cebu was favored over Ilocos	7
Ilocos	2	1	3	3	Ilocos was favored over Cebu	$11 + 9 + 3 = 23$
Number of senior citizens	11	9	7	3	<i>Ilocos will get 1 point</i>	

3. Between Cebu and Palawan

Cities	Ranking					
Cebu	3	4	2	4	Cebu was favored over Palawan	7
Palawan	1	2	4	1	Palawan was favored over Cebu	$11 + 9 + 3 = 23$
Number of senior citizens	11	9	7	3	<i>Palawan will get 1 point</i>	

4. Between Davao and Ilocos

Cities	Ranking					
Davao	4	3	1	2	Davao was favored over Ilocos	$7 + 3 = 10$
Ilocos	2	1	3	3	Ilocos was favored over Davao	$11 + 9 = 20$
Number of senior citizens	11	9	7	3	<i>Ilocos will get 1 point</i>	

5. Between Davao and Palawan

Cities	Ranking					
Davao	4	3	1	2	Davao was favored over Palawan	7
Palawan	1	2	4	1	Palawan was favored over Davao	$11 + 9 + 3 = 23$
Number of senior citizens	11	9	7	3	<i>Palawan will get 1 point</i>	

Cities	Ranking					
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Ilocos	2	1	3	3	Ilocos was favored over Palawan	7
Palawan	1	2	4	1	Palawan was favored over Ilocos	11 + 9 + 3 = 23
Number of senior citizens	11	9	7	3	<i>Palawan will get 1 point</i>	

6. Between Ilocos and Palawan

To summarize their points:

Cities	Points
Cebu	
Davao	1
Ilocos	1 + 1 = 2
Palawan	1 + 1 + 1 = 3

Therefore, by using the *pairwise comparison method*, **Palawan** is the most preferred tourist destination of the senior citizens

FAIRNESS CRITERIA

Among the voting methods presented, one will wonder which among these is the fairest; but taking into consideration the concept of what is fair, Kenneth J. Arrow had come out with various criteria for a fair voting system.

Majority Criterion. The candidate who receives a majority of the first-place votes is the winner.

Monotonicity Criterion. If candidate A wins an election, then candidate A will also win the election if the only change in the voters' preferences is that supporters of a different candidate change their votes to support candidate A.

Condorcet Criterion. A candidate who wins all head-head matchups should win an election when all candidates appear on the ballot.

Independence of Irrelevant Alternatives. If a candidate wins an election, the winner should remain the winner in any recount in which losing candidates withdraw from the race.

EXERCISES:

1. The BPSU Faculty Association is planning its annual team building activities and has asked its members to rank four different choices of venue in order of preferences. The result is given below:

Venue	Rankings				
beach	1	3	4	2	3
resort	2	2	3	1	2
conference hall	4	1	2	3	4
open field	3	4	1	4	1
Number of teachers	118	68	46	38	30

Determine where the team building shall be held using (a) plurality method; (b) plurality with elimination method; (c) borda count method and (c) pairwise comparison method

2. A group of BPSU students were asked to rank their favorite band according to their preferences for the University’s Kagitingan Awards. The table below shows the responses given by the students.

Name of Bands	Rankings				
Callalily	3	4	3	3	3
IV of Spades	2	1	1	2	2
Parokya Ni Edgar	1	2	2	1	4
Silent Sanctuary	4	3	2	4	1
Number of students	40	24	18	10	8

- Using the plurality method, determine if the winner satisfy the majority criterion,
 - Use plurality with elimination method to determine student’s favorite band.
 - Use the pairwise comparison method and check if the winner is the same as in letter b.
 - Does the winner satisfy the Condorcet criterion?
3. The Mathematics Society held an election for president. Instead of just voting for one candidate, the members of the society were asked to rank all four candidates in order of preference. The results are shown below.

Name of Candidate	Rankings					
Christian Aquino	3	2	1	2	4	3
Marcos Dela Cruz	1	3	3	4	1	4
Rodrigo Narciso	2	4	2	1	3	2
Shiela Sebastian	4	1	4	3	2	1
Number of members	53	45	41	36	31	27

Using the Borda count method, who should be the society’s president? Does the winner satisfy the majority criterion?

WEIGHTED VOTING SYSTEMS

There are times that votes are not treated equally in many voting systems; for example, in juries, if one voter votes “not guilty” then the result is “not guilty”. In case of companies, cooperatives and other business entities, if you have more shares, then your vote is weighted more heavily. In the US Electoral College, as well as in European Union and UN Security Council, larger member countries get more votes.

A *weighted voting system* of n voters is written $\{q: w_1, w_2 \dots w_n\}$ where q is the quota and w_1 through w_n represent the weights of each of the n voters.

Example No. 1: Given the voting system $\{20: 15, 10, 5\}$:

- In this voting system, there are three voters – the first voter has a weight of 15, the second 10 and the third is 5
- The quota is 20, so none of the voters can pass a bill by themselves.

Coalitions. A coalition is a set of voters each of whom votes the same way, either for or against a resolution.

Winning Coalition is a set of voters the sum of whose votes is greater than or equal to the quota.

Losing Coalition is a set of voters the sum of whose votes is less than the coalition.

Critical Voter is a voter who leaves a winning coalition and thereby turns it into a losing coalition.

Blocking Coalition is a collection of voters that can prevent a motion from passing by all voting “no,” even if every other voter votes “yes”. If the total number of votes is T and the quota is Q , then the coalition needs $T - Q + 1$ to be a blocking coalition.

The number of possible coalitions of n voters is
 $2^n - 1$

Example No. 2: Suppose a company has 2000 shares and owned by four proprietors Cheng (C), Lee (L), Tan (T) and Sy (S). Their respective shares are 800, 500, 300 and 400 shares. Half of the total shares is 1000, so the quota is 1001 and the weighted voting system for this company is $\{q: C, L, T, S\} = \{1001: 800, 500, 300, 400\}$.

- Determine the winning coalitions.
- For each winning coalition, determine the critical voters

Solution:

- a. The winning coalition must represent at least 1001 votes. The table below shows the winning coalitions:

Winning Coalition	Number of votes	Winning Coalition	Number of votes
{C, L}	1300	{C, L, S}	1700
{C, T}	1100	{L, T, S}	1200
{C, S}	1200	{C, L, T, S}	2000
{C, L, T}	1600		

- b. Critical voter is the one who makes the winning coalition a losing coalition when he leaves: For instance, the winning coalition {C, L}, if C leaves, the number of remaining votes is 500 which is not enough to pass a resolution; similarly if L leaves, the number of remaining votes is 800, still not enough to pass a resolution

Winning Coalition	Number of votes	Critical Voters	Winning Coalition	Number of votes	Critical Voters
{C, L}	1300	C, L	{C, L, S}	1700	C
{C, T}	1100	C, T	{L, T, S}	1200	none
{C, S}	1200	C, S	{C, L, T, S}	2000	none
{C, L, T}	1600	C			

Other Special Voters

Dictator is a voter that is a winning coalition by itself. It has a weight greater than or equal to quota.

Dummy voter is a voter whose vote does not matter or who is never a critical voter. When voters form a winning coalition with a dummy voter, it can be “safely” removed from the coalition without changing the result.

A voter has **veto power** if it is a blocking coalition by itself

More Examples:

1. {51: 60, 40}
 - In this system, the weight-60 voter is a dictator
2. {20: 15, 10, 5}
 - In this system, none of the voters is a dictator
3. {51: 26, 26, 22}
 - Any winning coalition that includes the weight-22 voter can safely remove it and still win

4. {21: 20, 15, 5}
 - The voter with weight 20 has veto power: if that voter votes “no,” then the motion cannot pass

5. Jury: {12: 1,1, 1, 1, 1,1, 1,1, 1, 1, 1, 1}
 - Since all 12 voters must vote in favor to pass the motion, each voter has *veto power*

6. {8: 5, 3, 1}
 - None of the voters is a dictator
 - The weight-1 voter is a dummy voter
 - The weight-5 and weight-3 voters have veto power

7. {9: 5, 3, 1}
 - Motions can only pass unanimously: none of the voters is a dictator, there are no dummies, and all of the voters have veto power

8. {6: 5, 3, 1}
 - None of the voters is a dictator
 - None of the voters is a dummy
 - The weight-5 voter has veto power

9. {51: 49, 48, 3}
 - None of the voters is a dictator
 - None of the voters is a dummy
 - None of the voters has veto power

10. {15: 15, 15, 10, 2, 1, 1}
 - The weight-15 voter is a dictator

Banzhaf Power Index (BPI)

This index was derived by John F. Banzhaf III in 1965 which is used to determine the power of a voter in a weighted voting system. If the index for voter A is less than the index for voter B. This means that A has fewer opportunities to form a winning coalition than does B.

The **Banzhaf Power Index** of a voter v , symbolized by $BPI(v)$, is given by

$$BPI(v) = \frac{\text{number of times voter } v \text{ is a critical voter}}{\text{number of any voter is a critical voter}}$$

Example No. 1: Suppose a weighted voting system for four voters A, B, C and D is given by {5: 3, 2, 2, 1}. Find the BPI of each of the four voters.

Solution: Determine first the winning coalitions with their respective number of votes and critical voters.

Winning Coalition	Number of votes	Critical Voters
{A, B}	$3 + 2 = 5$	A, B
{A, C}	$3 + 2 = 5$	A, C
{A, B, C}	$3 + 2 + 2 = 7$	A
{A, B, D}	$3 + 2 + 1 = 6$	A, B
{A, C, D}	$3 + 2 + 1 = 6$	A, C
{B, C, D}	$2 + 2 + 1 = 5$	B, C, D
{A, B, C, D}	$3 + 2 + 2 + 1 = 10$	none

Solving for BPI of each:

$$BPI(v) = \frac{\text{number of times voter } v \text{ is a critical voter}}{\text{number of any voter is a critical voter}}$$

$$BPI(A) = \frac{5}{12} \approx 0.42 \quad BPI(B) = \frac{3}{12} \approx 0.25$$
$$BPI(C) = \frac{3}{12} \approx 0.25 \quad BPI(D) = \frac{1}{12} \approx 0.08$$

This shows that voter A has greater power than voter B or voter C; voter B and C has equal power while voter D has the least power.

Example No. 2: Consider the weighted voting system $\{60: 4, 56, 58\}$. Compute for the Banzhaf power index for each voter.

Solution:

Winning Coalition	Number of votes	Critical Voters
$\{A, B\}$	$4 + 56 = 60$	A, B
$\{A, C\}$	$4 + 58 = 62$	A, C
$\{B, C\}$	$56 + 58 = 114$	B, C
$\{A, B, C\}$	$4 + 56 + 58 = 118$	A, B, C

Solving for BPI of each:

$$BPI(v) = \frac{\text{number of times voter } v \text{ is a critical voter}}{\text{number of any voter is a critical voter}}$$

$$BPI(A) = \frac{3}{9} \approx 0.33 \quad BPI(B) = \frac{3}{9} \approx 0.33 \quad BPI(C) = \frac{3}{9} \approx 0.33$$

Notice that although voter A has a weight of 4 which is lower than voter B and voter C, solving for their BPI shows that they have equal power. It appears that voter A appears the same number of times as critical voter as voter B and voter C.

EXERCISES:

1. In the following exercises, the weighted voting system for voters A, B, C, \dots , the system are given in the form $\{q: w_1, w_2, \dots, w_n\}$ where w_1 is the weight of voter A, w_2 is the weight of voter B and so on. In each of the following, answer the following questions:
 - a. What is the quota?
 - b. How many voters are in the system?
 - c. What is the weight of voter C?
 - d. What is the weight of the coalition $\{B, C\}$?
 - e. Is $\{A, B, D\}$ a winning coalition?
 - f. Which voters are critical voters in the coalition $\{B, C, D\}$?
 - g. How many coalition can be formed?
 - h. How many winning coalitions consist of exactly three voters?

Given	a	b	c	d	e	f	g	h
{16: 8, 7, 4, 2, 1}								
{25: 15, 10, 5, 8}								
{111: 100, 80, 20, 10, 1}								
{3: 1, 1, 1, 1}								
{18: 18, 7, 3, 3, 1}								

2. Determine the Banzhaf power index of voter D in each of the following system in Number 1.
3. Which, if any of the voting systems in Number 1 is
 - a. a dictatorship?
 - b. a veto power system?
 - c. one-person, one vote system?

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