

# DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS



# TRANSCENDENTAL FUNCTIONS

Kinds of transcendental functions:

1. logarithmic and exponential functions
2. trigonometric and inverse trigonometric functions
3. hyperbolic and inverse hyperbolic functions

Note:

Each pair of functions above is an inverse to each other.



# The INVERSE TRIGONOMETRIC FUNCTIONS

## *Definition s and Properties of Inverse Trigonometric Functions*

- *if  $y$  is a function of  $x$  determined by the relation  $\sin y = x \rightarrow y$  is called the inverse sine function of  $x$  denoted by  $y = \arcsin x$  or  $y = \sin^{-1} x$*
- *This also mean  $y$  is the angle whose sine is  $x$ .*

*In general, these are the following definition s :*

$$y = \sin^{-1} x \quad \implies \quad \text{if } \sin y = x \quad \text{where : } -\pi/2 \leq y \leq \pi/2$$

$$y = \cos^{-1} x \quad \implies \quad \text{if } \cos y = x \quad \text{where : } 0 \leq y \leq \pi$$

$$y = \tan^{-1} x \quad \implies \quad \text{if } \tan y = x \quad \text{where : } -\pi/2 < y < \pi/2$$

$$y = \cot^{-1} x \quad \implies \quad \text{if } \cot y = x \quad \text{where : } 0 < y < \pi$$

$$y = \sec^{-1} x \quad \implies \quad \text{if } \sec y = x \quad \text{where : } \begin{array}{ll} 0 \leq y < \pi/2 & \text{if } x \geq 1 \\ \text{or} & \\ \pi/2 < y \leq \pi & \text{if } x \leq -1 \end{array}$$

$$y = \csc^{-1} x \quad \implies \quad \text{if } \csc y = x \quad \text{where : } \begin{array}{ll} 0 < y \leq \pi/2 & \text{if } x \geq 1 \\ \text{or} & \\ -\frac{\pi}{2} \leq y < 0 & \text{if } x \leq -1 \end{array}$$



# DIFFERENTIATION FORMULAS

*In finding the derivative of  $y = \sin^{-1} x$ , we use the definition of inverse trigonometric*

*function  $\rightarrow \sin y = x$  where  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$*

*Differentiating with respect to  $y$ :*

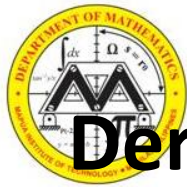
$$\cos y = \frac{dx}{dy} \text{ or } \frac{dy}{dx} = \frac{1}{\cos y}$$

*from the identity :  $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$*

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \text{ but } y = \sin^{-1} x \rightarrow \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

$$\text{Therefore } \frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}$$

*In similar manner we can derive the formulas for the other trigonometric functions.*



# Derivatives of Inverse Trigonometric Functions

*Differentiation formulas for inverse trigonometric functions :*

$$1. \frac{d}{dx} (\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$2. \frac{d}{dx} (\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$3. \frac{d}{dx} (\tan^{-1} u) = \frac{1}{1+u^2} \frac{du}{dx}$$

$$4. \frac{d}{dx} (\cot^{-1} u) = -\frac{1}{1+u^2} \frac{du}{dx}$$

$$5. \frac{d}{dx} (\sec^{-1} u) = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$$

$$6. \frac{d}{dx} (\csc^{-1} u) = -\frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$$



## EXAMPLES:

A. Find the derivative of each of the following functions and simplify the result:

1.  $f(x) = \sin^{-1} x^3$

$$f'(x) = \frac{1}{\sqrt{1-(x^3)^2}} (3x^2)$$

$$f'(x) = \frac{3x^2}{\sqrt{1-x^6}}$$

$$f'(x) = \frac{3x^2}{\sqrt{1-x^6}} \cdot \frac{\sqrt{1-x^6}}{\sqrt{1-x^6}}$$

$$f'(x) = \frac{3x^2 \sqrt{1-x^6}}{1-x^6}$$

2.  $f(x) = \cos^{-1}(3x)$

$$f'(x) = -\frac{1}{\sqrt{1-(3x)^2}} (3)$$

$$f'(x) = \frac{-3}{\sqrt{1-9x^2}}$$

$$f'(x) = \frac{-3}{\sqrt{1-9x^2}} \cdot \frac{\sqrt{1-9x^2}}{\sqrt{1-9x^2}}$$

$$f'(x) = \frac{-3\sqrt{1-9x^2}}{1-9x^2}$$



$$3. \quad y = \sec^{-1}(2x^2)$$

$$y' = \frac{1}{2x^2 \sqrt{(2x^2)^2 - 1}} (4x)$$

$$y' = \frac{2}{x \sqrt{4x^4 - 1}}$$

$$y' = \frac{2}{x \sqrt{4x^4 - 1}} \cdot \frac{\sqrt{4x^4 - 1}}{\sqrt{4x^4 - 1}}$$

$$y' = \frac{2\sqrt{4x^4 - 1}}{x(4x^4 - 1)}$$

$$4. \quad y = 2 \cos^{-1} \sqrt{x}$$

$$y' = 2 \cdot \frac{-1}{\sqrt{1 - (\sqrt{x})^2}} \cdot \left( \frac{1}{2\sqrt{x}} \right)$$

$$y' = \frac{-1}{\sqrt{1-x} \cdot \sqrt{x}} = \frac{-1}{\sqrt{x(1-x)}}$$

$$y' = \frac{-1}{\sqrt{x(1-x)}} \cdot \frac{\sqrt{x(1-x)}}{\sqrt{x(1-x)}}$$

$$y' = \frac{-\sqrt{x(1-x)}}{x(1-x)}$$



$$h(x) = \frac{1}{2} \sin^{-1}(2e^x)$$

$$h'(x) = \frac{1}{2} \cdot \frac{2e^x}{\sqrt{1-(2e^x)^2}} = \frac{e^x}{\sqrt{1-4e^{2x}}} \cdot \frac{\sqrt{1-4e^{2x}}}{\sqrt{1-4e^{2x}}} = \frac{e^x \sqrt{1-4e^{2x}}}{1-4e^{2x}}$$

6.  $g(t) = \sec^{-1} 5t + \csc^{-1} 5t$

$$g'(t) = \frac{1}{5t\sqrt{25t^2-1}}(5) + \frac{(-1)}{5t\sqrt{25t^2-1}}(5)$$

$$g'(t) = 0$$

7.  $g(x) = \cot^{-1} \frac{2}{x}$

$$g'(x) = \frac{-1}{1 + \left(\frac{2}{x}\right)^2} \left(\frac{-2}{x^2}\right) = \frac{2}{\left(1 + \frac{4}{x^2}\right) \cdot x^2} \rightarrow g'(x) = \frac{2}{x^2 + 4}$$



$$f(x) = x^2 \tan^{-1}(3x)$$

$$f(x) = x^2 \left( \frac{1}{1+(3x)^2} \cdot 3 \right) + \tan^{-1} 3x \cdot 2x$$

$$f(x) = x \left( \frac{3x}{1+9x^2} + 2 \tan^{-1} 3x \right)$$

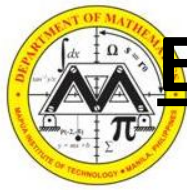
$$f(x) = x \left( \frac{3x + 2(1+9x^2)(\tan^{-1} 3x)}{1+9x^2} \right)$$

$$9. \quad y = \text{Sec}^{-1} \left( \csc \frac{5}{x} \right)$$

$$y' = \frac{-\csc \frac{5}{x} \cot \frac{5}{x} \left[ -\frac{5}{x^2} \right]}{\csc \frac{5}{x} \sqrt{\left( \csc \frac{5}{x} \right)^2 - 1}}$$

$$\text{but, } \sqrt{\left( \csc \frac{5}{x} \right)^2 - 1} = \sqrt{\left( \cot \frac{5}{x} \right)^2} = \cot \frac{5}{x}$$

$$y' = \frac{5}{x^2}$$



## EXERCISES:

A. Find the derivative and simplify the result.

1.  $g(x) = 3 \tan^{-1} 3x$

2.  $y = x \sin^{-1} 2x + \frac{1}{2} \cot^{-1} \sqrt{x}$

3.  $f(x) = \sin^{-1} \frac{4}{x^3}$

4.  $y = (\operatorname{arccsc} 2x)^4$

5.  $G(x) = 5x^2 \operatorname{Cos}^{-1} 2x$

6.  $y = \cos^{-1}(\sin x)$

7.  $F(x) = \frac{\cot^{-1} 3x^2}{9x}$

8.  $y = \sin^{-1}(\tan^{-1} 3x)$

9.  $h(x) = \sin^{-1} 3x - 6x \sec^{-1} x^2$

10.  $y = \frac{7x^5}{\cot^{-1} 5x^2}$



B. Find the derivative and simplify the result.

1.  $g(t) = t\sqrt{4-t^2} + 4 \arcsin \frac{t}{2}$

6.  $h(y) = y^5 (\arccos y)^5$

2.  $y = -\cos^{-1} x^2$

7.  $y = \arcsin \left( \frac{x}{\sqrt{x^2 + 4}} \right)$

3.  $f(z) = z^4 \arcsin 3z$

8.  $F(y) = \arctan \left( \frac{1+y}{1-y} \right)$

4.  $y = 7 \cos^{-1} (\sqrt{2x})$

9.  $y = \tan^{-1} 4x + \cos^{-1} 4x$

5.  $y = \tan^{-1} (1-7x)$

10.  $H(x) = \frac{\sqrt{x+4}}{\tan^{-1} 4x}$



# APPLICATIONS

1. A balloon leaving the ground 60 ft. from an observer rises 10 ft/sec. How fast is the angle of elevation of the line of sight decreasing after 8 seconds?
2. A ladder 10 ft. long leans against a wall. Find the rate how fast is the angle  $\theta$  between the ladder and the ground changing at the given moment when the ladder is 6 ft. away from the wall and sliding along the ground away from the wall at the rate of 2 ft/sec.
3. A boat is being pulled into a dock by a rope at the rate of 2 ft/sec and the rope passes through a ring on the bow of the boat. The dock is 8 ft. higher than the bow ring. Find the rate of change of the angle  $\theta$  when there is 10 ft. of rope out.



# APPLICATIONS

4. A statue 5 ft. tall stands on a pedestal 9 ft. high. If an observer's eye is 5 ft. above the ground, how far from the pedestal should he stand so that the angle subtended in his eye by the statue will be a maximum?
5. A kite is 60 ft. high with 100 ft. of cord out. If the kite is moving horizontally 4 mph directly away from the boy flying it, find the rate of change of the angle of elevation of the cord.
6. A ship moving at 8 mph sails E for 2 hrs. then turns N  $30^\circ$  W. A search light placed at the starting point follows the ship. Find how fast the light is rotating (a) 3 hours after the start, (b) just after the turn.