DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

TRANSCENDENTAL FUNCTIONS

Kinds of transcendental functions:

1. logarithmic and exponential functions
2. trigonometric and inverse trigonometric functions
3. hyperbolic and inverse hyperbolic functions

Note:
Each pair of functions above is an inverse to each other.

The INVERSE TRIGONOMETRIC FUNCTIONS

Recall

Definitions and Properties of Inverse Trigonometric Functions

- *if y is a function of x determined by the relation*

  $\sin y = x \rightarrow y$ *is called the inverse sine function of x*

  denoted by $y = \arcsin x$ *or* $y = \sin^{-1} x$

- *This also mean y is the angle whose sine is x.*

In general, these are the following definitions:

$y = \sin^{-1} x \quad \Longrightarrow \quad$ if $\sin y = x \quad$ where : $\quad -\pi/2 \le y \le \pi/2$

$y = \cos^{-1} x \quad \Longrightarrow \quad$ if $\cos y = x \quad$ where : $\quad 0 \le y \le \pi$

$y = \tan^{-1} x \quad \Longrightarrow \quad$ if $\tan y = x \quad$ where : $\quad -\pi/2 < y < \pi/2$

$y = \cot^{-1} x \quad \Longrightarrow \quad$ if $\cot y = x \quad$ where : $\quad 0 < y < \pi$

$y = \sec^{-1} x \quad \Longrightarrow \quad$ if $\sec y = x \quad$ where : $\quad 0 \le y < \pi/2 \quad$ if $x \ge 1$

or $\quad \pi/2 < y \le \pi \quad$ if $x \le -1$

$y = \csc^{-1} x \quad \Longrightarrow \quad$ if $\csc y = x \quad$ where : $\quad 0 < y \le \pi/2 \quad$ if $x \ge 1$

or $\quad -\frac{\pi}{2} \le y < 0 \quad$ if $x \le -1$

DIFFERENTIATION FORMULAS

In finding the derivative of $y = \sin^{-1} x$, *we use the definition of inverse trigonomet ric function* $\rightarrow \sin y = x$ *where* $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Differentia ting with respect to y :

$$\cos y = \frac{dx}{dy} \text{ or } \frac{dy}{dx} = \frac{1}{\cos y}$$

from the identity : $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \text{ but } y = \sin^{-1} x \rightarrow \frac{d\left(\sin^{-1} x\right)}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

$$\textbf{Therefore} \frac{d}{dx}\left(\sin^{-1} u\right) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}$$

In similar manner we can derive the formulas for the other trigonomet ric functions.

Derivatives of Inverse Trigonometric Functions

Differenti ation formulas for inverse trigonomet ric functions :

1. $\frac{d}{dx}\left(\sin^{-1} u\right) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

2. $\frac{d}{dx}\left(\cos^{-1} u\right) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

3. $\frac{d}{dx}\left(\tan^{-1} u\right) = \frac{1}{1+u^2}\frac{du}{dx}$

4. $\frac{d}{dx}\left(\cot^{-1} u\right) = -\frac{1}{1+u^2}\frac{du}{dx}$

5. $\frac{d}{dx}\left(\sec^{-1} u\right) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$

6. $\frac{d}{dx}\left(\csc^{-1} u\right) = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$

EXAMPLES:

A. Find the derivative of each of the following functions and simplify the result:

1. $f(x) = \sin^{-1} x^3$

$$f(x) = \frac{1}{\sqrt{1-(x^3)^2}}(3x^2)$$

$$f'(x) = \frac{3x^2}{\sqrt{1-x^6}}$$

$$f'(x) = \frac{3x^2}{\sqrt{1-x^6}} \bullet \frac{\sqrt{1-x^6}}{\sqrt{1-x^6}}$$

$$f'(x) = \frac{3x^2\sqrt{1-x^6}}{1-x^6}$$

2. $f(x) = \cos^{-1}(3x)$

$$f'(x) = -\frac{1}{\sqrt{1-(3x)^2}}(3)$$

$$f'(x) = \frac{-3}{\sqrt{1-9x^2}}$$

$$f'(x) = \frac{-3}{\sqrt{1-9x^2}} \bullet \frac{\sqrt{1-9x^2}}{\sqrt{1-9x^2}}$$

$$f'(x) = \frac{-3\sqrt{1-9x^2}}{1-9x^2}$$

3. $y = \sec^{-1}\left(2x^2\right)$

$$y' = \frac{1}{2x^2\sqrt{\left(2x^2\right)^2 - 1}}(4x)$$

$$y' = \frac{2}{x\sqrt{4x^4 - 1}}$$

$$y' = \frac{2}{x\sqrt{4x^4 - 1}} \bullet \frac{\sqrt{4x^4 - 1}}{\sqrt{4x^4 - 1}}$$

$$y' = \frac{2\sqrt{4x^4 - 1}}{x\left(4x^4 - 1\right)}$$

4. $y = 2\cos^{-1}\sqrt{x}$

$$y' = 2 \cdot \frac{-1}{\sqrt{1 - \left(\sqrt{x}\right)^2}} \cdot \left(\frac{1}{2\sqrt{x}}\right)$$

$$y' = \frac{-1}{\sqrt{1 - x} \cdot \sqrt{x}} = \frac{-1}{\sqrt{x(1 - x)}}$$

$$y' = \frac{-1}{\sqrt{x(1 - x)}} \bullet \frac{\sqrt{x(1 - x)}}{\sqrt{x(1 - x)}}$$

$$y' = \frac{-\sqrt{x(1 - x)}}{x(1 - x)}$$

$$h(x)=\frac{1}{2}\sin^{-1}\left(2e^{x}\right)$$

$$h'(x)=\frac{1}{2}\cdot\frac{2e^{x}}{\sqrt{1-\left(2e^{x}\right)^{2}}}=\frac{e^{x}}{\sqrt{1-4e^{2x}}}\bullet\frac{\sqrt{1-4e^{2x}}}{\sqrt{1-4e^{2x}}}=\frac{e^{x}\sqrt{1-4e^{2x}}}{1-4e^{2x}}$$

6. $g(t)=\sec^{-1}5t+\csc^{-1}5t$

$$g'(t)=\frac{1}{5t\sqrt{25t^{2}-1}}(5)+\frac{(-1)}{5t\sqrt{25t^{2}-1}}(5)$$

$$g'(t)=0$$

7. $g(x)=\cot^{-1}\frac{2}{x}$

$$g'(x)=\frac{-1}{1+\left(\frac{2}{x}\right)^{2}}\left(\frac{-2}{x^{2}}\right)=\frac{2}{\left(1+\frac{4}{x^{2}}\right)\cdot x^{2}}\quad\rightarrow\quad g'(x)=\frac{2}{x^{2}+4}$$

$$f(x) = x^2 \tan^{-1}(3x)$$

$$f(x) = x^2\left(\frac{1}{1+(3x)^2} \bullet 3\right) + \tan^{-1} 3x \bullet 2x$$

$$f(x) = x\left(\frac{3x}{1+9x^2} + 2\tan^{-1} 3x\right)$$

$$f(x) = x\left(\frac{3x + 2\left(1+9x^2\right)\left(\tan^{-1} 3x\right)}{1+9x^2}\right)$$

9. $y = Sec^{-1}(\csc\frac{5}{x})$

$$y' = \frac{-\csc\frac{5}{x}\cot\frac{5}{x}\left[-\frac{5}{x^2}\right]}{\csc\frac{5}{x}\sqrt{\left(\csc\frac{5}{x}\right)^2 - 1}} \qquad but, \quad \sqrt{\left(\csc\frac{5}{x}\right)^2 - 1} = \sqrt{\left(\cot\frac{5}{x}\right)^2} = \cot\frac{5}{x}$$

$$y' = \frac{5}{x^2}$$

EXERCISES:

A. Find the derivative and simplify the result.

1. $g(x) = 3\tan^{-1} 3x$

2. $y = x\sin^{-1} 2x + \frac{1}{2}\cot^{-1}\sqrt{x}$

3. $f(x) = \sin^{-1}\frac{4}{x^3}$

4. $y = (\text{arc}\csc 2x)^4$

5. $G(x) = 5x^2 \text{Cos}^{-1} 2x$

6. $y = \cos^{-1}(\sin x)$

7. $F(x) = \frac{\cot^{-1} 3x^2}{9x}$

8. $y = \sin^{-1}(\tan^{-1} 3x)$

9. $h(x) = \sin^{-1} 3x - 6x\sec^{-1} x^2$

10. $y = \frac{7x^5}{\cot^{-1} 5x^2}$

B. Find the derivative and simplify the result.

1. $g(t)=t\sqrt{4-t^2}+4\arcsin\frac{t}{2}$

2. $y=-\cos^{-1}x^2$

3. $f(z)=z^4\operatorname{arcsec}3z$

4. $y=7\cos^{-1}\left(\sqrt{2x}\right)$

5. $y=\tan^{-1}(1-7x)$

6. $h(y)=y^5(\arccos y)^5$

7. $y=\arcsin\left(\frac{x}{\sqrt{x^2+4}}\right)$

8. $F(y)=\arctan\left(\frac{1+y}{1-y}\right)$

9. $y=\tan^{-1}4x+\cos^{-1}4x$

10. $H(x)=\frac{\sqrt{x+4}}{\tan^{-1}4x}$

APPLICATIONS

1. A balloon leaving the ground 60 ft. from an observer rises 10 ft/sec. How fast is the angle of elevation of the line of sight decreasing after 8 seconds?
2. A ladder 10 ft. long leans against a wall. Find the rate how fast is the angle θ between the ladder and the ground changing at the given moment when the ladder is 6 ft. away from the wall and sliding along the ground away from the wall at the rate of 2 ft/sec.
3. A boat is being pulled into a dock by a rope at the rate of 2 ft/sec and the rope passes through a ring on the bow of the boat. The dock is 8 ft. higher than the bow ring. Find the rate of change of the angle θ when there is 10 ft. of rope out.

APPLICATIONS

4. A statue 5 ft. tall stands on a pedestal 9 ft. high. If an observer's eye is 5 ft. above the ground, how far from the pedestal should he stand so that the angle subtended in his eye by the statue will be a maximum?
5. A kite is 60 ft. high with 100 ft. of cord out. If the kite is moving horizontally 4 mph directly away from the boy flying it, find the rate of change of the angle of elevation of the cord.
6. A ship moving at 8 mph sails E for 2 hrs. then turns N 30° W. A search light placed at the starting point follows the ship. Find how fast the light is rotating (a) 3 hours after the start, (b) just after the turn.