Chapter 4 - Trigonometric and Inverse Trigonometric Functions

Differentiation of Trigonometric Functions

Trigonometric identities and formulas are basic requirements for this section. If *u* is a function of *x*, then

1. $\frac{d}{dx}(\sin\ u) = \cos\ u \frac{du}{dx}$
2. $\frac{d}{dx}(\cos\ u) = -\sin\ u \frac{du}{dx}$
3. $\frac{d}{dx}(\tan\ u) = \sec^2\ u \frac{du}{dx}$
4. $\frac{d}{dx}(\cot\ u) = -\csc^2\ u \frac{du}{dx}$
5. $\frac{d}{dx}(\sec\ u) = \sec\ u\ \tan\ u \frac{du}{dx}$
6. $\frac{d}{dx}(\csc\ u) = -\csc\ u\ \cot\ u \frac{du}{dx}$

Differentiation of Inverse Trigonometric Functions

In the formula below, *u* is any function of *x*.

1. $\frac{d}{dx} \arcsin\ u = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}$
2. $\frac{d}{dx} \arccos\ u = -\frac{\frac{du}{dx}}{\sqrt{1-u^2}}$
3. $\frac{d}{dx} \arctan\ u = \frac{\frac{du}{dx}}{1+u^2}$
4. $\frac{d}{dx} \text{arccot}\, u = -\frac{\frac{du}{dx}}{1+u^2}$

$$\frac{d}{dx}\operatorname{arcsec} u = \frac{\frac{du}{dx}}{u\sqrt{u^2-1}}$$
5.

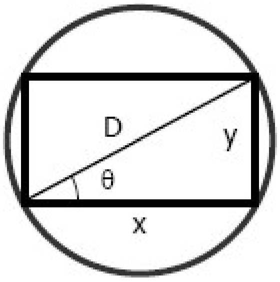
$$\frac{d}{dx}\operatorname{arccsc} u = -\frac{\frac{du}{dx}}{u\sqrt{u^2-1}}$$
6.

Maxima and Minima Using Trigonometric Functions

Problem 01
Find the shape of the rectangle of maximum perimeter inscribed in a circle.

Solution 01

Perimeter of rectangle,
$P = 2x + 2y$



Where:
$x = D\cos\theta$

$y = D\sin\theta$

$P = 2D\cos\theta + 2D\sin\theta$

$$\frac{dP}{d\theta} = -2D\sin\theta + 2D\cos\theta = 0$$

$-\sin\theta + \cos\theta = 0$

$\sin\theta = \cos\theta$

$$\frac{\sin\theta}{\cos\theta} = 1$$

$\tan\theta = 1$

$\theta = 45^\circ$

$x = D\cos 45^\circ = 0.707D$

$y = D\sin 45^\circ = 0.707D$

$x = y$ (square)

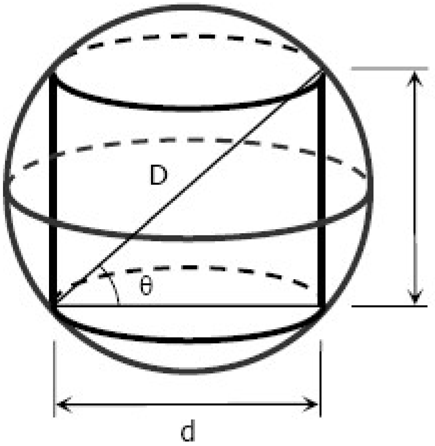
answer

See also the solution using algebraic function.

Problem 02
A cylinder is inscribed in a given sphere. Find the shape of the cylinder if its convex surface area is a maximum.

Solution 02

Convex surface area of cylinder,
$A = \pi dh$



Where:
$d = D\cos\theta$

$h = D\sin\theta$

$A = \pi(D\cos\theta)(D\sin\theta)$

$A = D^2\pi\cos\theta\sin\theta$

$\dfrac{dA}{d\theta} = D^2\pi(\cos^2\theta - \sin^2\theta) = 0$

$\sin^2\theta = \cos^2\theta$

$\tan^2\theta = 1$

$\theta = 45^\circ$

$d = D\cos 45^\circ = 0.707D$

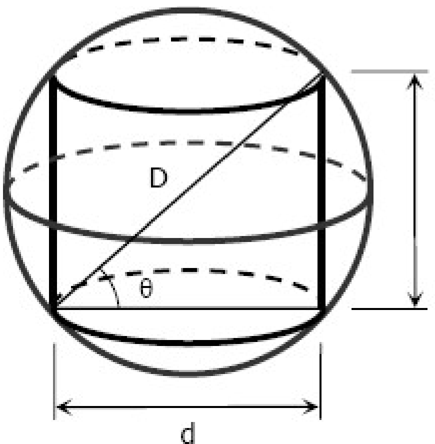
$h = D\sin 45^\circ = 0.707D$

$\text{diameter} = \text{height}$

answer

Problem 3
Find the weight of the heaviest circular cylinder can be cut from a 16-lb shot.

Solution 03



A shot is in the form of a sphere and the cylinder is the cylinder of maximum. From the figure)This is also the figure used in Solution 02:

$V_c = \frac{1}{4}\pi d^2 h$

Where:
$d = D\cos\theta$

$h = D\sin\theta$

Thus,
$V_c = \frac{1}{4}\pi(D\cos\theta)^2(\sin\theta)$

$V_c = \frac{1}{4}D^3\pi\cos^2\theta\sin\theta$

$\frac{dV_c}{d\theta} = \frac{1}{4}D^3\pi[\,\cos^2\theta(\cos\theta) + \sin\theta(-2\cos\theta\sin\theta)\,]$

$\frac{dV_c}{d\theta} = \frac{1}{4}D^3\pi(\cos^3\theta - 2\sin^2\theta\cos\theta) = 0$

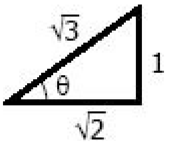
$2\sin^2\theta\cos\theta = \cos^3\theta$

$2\sin^2\theta = \cos^2\theta$

$\frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{2}$

$\tan^2\theta = \frac{1}{2}$

$\tan\theta = \frac{1}{\sqrt{2}}$



$\cos\theta = \frac{\sqrt{2}}{\sqrt{3}}$

$\sin\theta = \frac{1}{\sqrt{3}}$

$V_c = \frac{1}{4}D^3\pi\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)$

$V_c = \frac{1}{6\sqrt{3}}\pi D^3$
→ Maximum volume of cylinder

Volume of shot (sphere):
$V_s = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi(D/2)^3$

$V_s = \frac{1}{6}\pi D^3$

Weight is proportional to the volume, so
$\frac{W_c}{V_c} = \frac{W_s}{V_s}$

$$W_c = \frac{W_s}{V_s} \times V_c$$

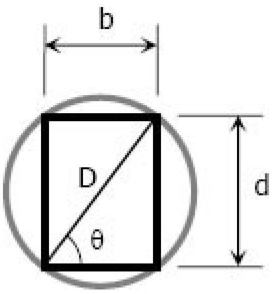
$$W_c = \frac{16}{\frac{1}{6}\pi D^3} \times \frac{1}{6\sqrt{3}}\pi D^3$$

$$W_c = 9.24 \text{ lb}$$

answer

Problem 4
The stiffness of a rectangular beam is proportional to the breadth and the cube of the depth. Find the shape of the stiffest beam that can be cut from a log of a given size.

Solution 4



Stiffness,

$$k = bd^3$$

Where:

$$b = D\cos\theta$$

$$d = D\sin\theta$$

$$k = D^4 \cos\theta \, \sin^3\theta$$

$$\frac{dk}{d\theta} = D^4(3\cos^2\theta \, \sin^2\theta - \sin^4\theta) = 0$$

$$3\cos^2\theta - \sin^2\theta = 0$$

$$\sin^2\theta = 3\cos^2\theta$$

$$\tan^2\theta = 3$$

$$\tan\theta = \sqrt{3}$$

$$\theta = 60^\circ$$

$$b = D\cos 60^\circ = \tfrac{1}{2}D$$

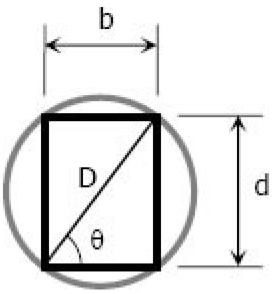
$$d = D\sin 60^\circ = \tfrac{1}{2}\sqrt{3}D$$

$$\text{depth} = \sqrt{3} \times \text{breadth}$$

answer

Problem 5
The strength of rectangular beam is proportional to the breadth and the square of the depth. Find the shape of the strongest beam that can be cut from a log of given size.

Solution 5



Strength,

$S = bd^2$

Where:

$b = D\cos\theta$

$d = D\sin\theta$

$S = D^3\cos\theta\sin^2\theta$

$S = D^3\cos\theta(1-\cos^2\theta)$

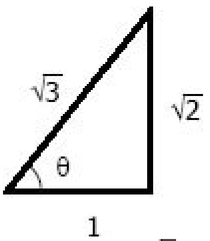
$S = D^3(\cos\theta - \cos^3\theta)$

$\dfrac{dS}{d\theta} = D^3(-\sin\theta + 3\cos^2\theta\sin\theta) = 0$

$-1 + 3\cos^2\theta = 0$

$\cos^2\theta = \frac{1}{3}$

$\cos\theta = \frac{1}{\sqrt{3}}$



$b = D\cos\theta = \frac{1}{\sqrt{3}}D$

$d = D\sin\theta = \dfrac{1}{\sqrt{3}}\sqrt{2}\,D$

$\text{depth} = \sqrt{2}\times\ \text{breadth}$

answer

Problem 06

A trapezoidal gutter is to be made, from a strip of metal 22 inches wide by bending up the edges. If the base is 14 inches wide, what width across the top gives the greatest carrying capacity.

Solution 06

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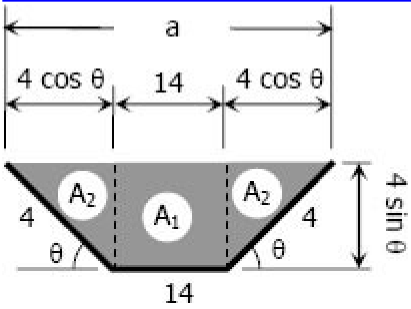


$A_1 = 14(4\sin\theta) = 56\sin\theta$

$A_2 = \frac{1}{2}(4\cos\theta)(4\sin\theta) = 8\cos\theta\sin\theta$

$A = A_1 + 2A_2$

$A = 56\sin\theta + 2(8\cos\theta\sin\theta)$

$$\frac{dA}{d\theta} = 56\cos\theta + 16(\cos^2\theta - \sin^2\theta) = 0$$

$$7\cos\theta + 2[\cos^2\theta - (1 - \cos^2\theta)] = 0$$

$$4\cos^2\theta + 7\cos\theta - 2 = 0$$

$$(4\cos\theta - 1)(\cos\theta + 2) = 0$$

for
$$4\cos\theta - 1 = 0$$

$$\cos\theta = 1/4$$

for
$$\cos\theta + 2 = 0$$

$$\cos\theta = -2$$
→ (meaningless)

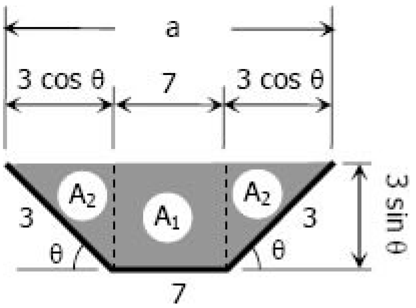
use
$$\cos\theta = 1/4$$

$$a = 14 + 2(4\cos\theta) = 14 + 8(1/4)$$

$$a = 16 \text{ inches}$$
answer

Problem 07
Solve Problem 06, if the strip is 13 inches wide and the base width 7 inches.

Solution 07



$$A_1 = 7(3\sin\theta) = 21\sin\theta$$

$$A_2 = \tfrac{1}{2}(3\cos\theta)(3\sin\theta) = 4.5\cos\theta\sin\theta$$

$$A = A_1 + 2A_2$$

$$A = 21\sin\theta + 2(4.5\cos\theta\sin\theta)$$

$$\frac{dA}{d\theta} = 21\cos\theta + 9(\cos^2\theta - \sin^2\theta) = 0$$

$$7\cos\theta + 3[\cos^2\theta - (1 - \cos^2\theta)] = 0$$

$$6\cos^2\theta + 7\cos\theta - 3 = 0$$

$(3\cos\theta - 1)(2\cos\theta + 3) = 0$

for
$3\cos\theta - 1 = 0$

$\cos\theta = 1/3$

for
$2\cos\theta + 3 = 0$

$\cos\theta = -3/2$
→ (meaningless)

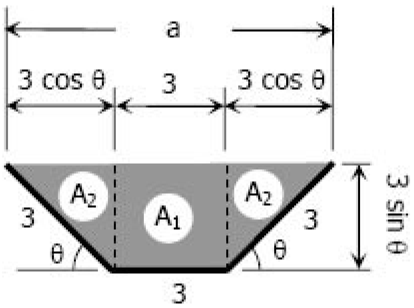
$\cos\theta = 1/3$
use

$a = 7 + 2(3\cos\theta) = 7 + 6(1/3)$

$a = 9 \text{ inches}$
answer

Problem 08
Solve Problem 06 if the strip is 9 inches wide and the base width 3 inches.

Solution 08



$A_1 = 3(3\sin\theta) = 9\sin\theta$

$A_2 = \frac{1}{2}(3\cos\theta)(3\sin\theta) = 4.5\cos\theta\sin\theta$

$A = A_1 + 2A_2$

$A = 9\sin\theta + 2(4.5\cos\theta\sin\theta)$

$\dfrac{dA}{d\theta} = 9\cos\theta + 9(\cos^2\theta - \sin^2\theta) = 0$

$\cos\theta + [\,\cos^2\theta - (1 - \cos^2\theta)\,] = 0$

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

for
$2\cos\theta - 1 = 0$

$\cos\theta = 1/2$

for
$\cos\theta + 1 = 0$

$\cos\theta = -1$
→ (meaningless)

$\cos\theta = 1/2$
use

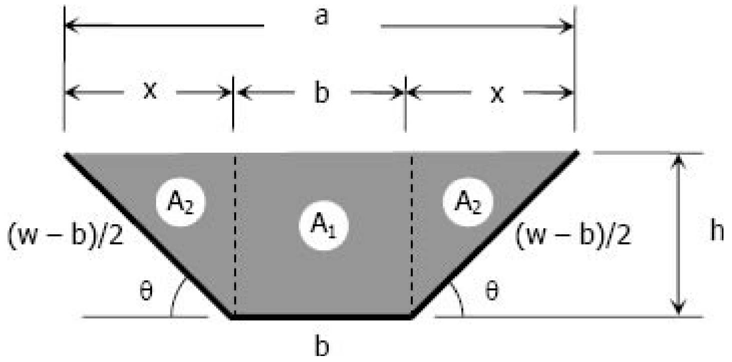
$a = 3 + 2(3\cos\theta) = 3 + 6(1/2)$

$a = 6 \ \ \text{inches}$
answer

Problem 09

Solve Problem 06, if the strip width is w and the base width b.

Solution 09



$x = \frac{1}{2}(w - b)\cos\theta$

$h = \frac{1}{2}(w - b)\sin\theta$

$A_1 = bh$

$A_1 = \frac{1}{2}b(w - b)\sin\theta$

$A_2 = \frac{1}{2}xh$

$A_2 = \frac{1}{2}[\,\frac{1}{2}(w - b)\cos\theta\,][\,\frac{1}{2}(w - b)\sin\theta\,]$

$A_2 = \frac{1}{8}(w - b)^2\sin\theta\cos\theta$

$A = A_1 + 2A_2$

$A = \frac{1}{2}b(w - b)\sin\theta + 2[\,\frac{1}{8}(w - b)^2\sin\theta\cos\theta\,]$

$$\frac{dA}{d\theta} = \tfrac{1}{2}b(w-b)\cos\theta + \tfrac{1}{4}(w-b)^2(\cos^2\theta - \sin^2\theta)$$

$$2b\cos\theta + (w-b)[\cos^2\theta - (1-\cos^2\theta)] = 0$$

$$2(w-b)\cos^2\theta + 2b\cos\theta - (w-b) = 0$$

By quadratic formula:
A = 2(w - b); B = 2b; C = -(w - b)

$$\cos\theta = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\cos\theta = \frac{-2b \pm \sqrt{(2b)^2 - 4[2(w-b)][-(w-b)]}}{2[2(w-b)]}$$

$$\cos\theta = \frac{-2b \pm \sqrt{4b^2 + 8(w-b)^2}}{4(w-b)}$$

$$\cos\theta = \frac{-2b \pm 2\sqrt{b^2 + 4(w-b)^2}}{4(w-b)}$$

$$\cos\theta = \frac{-b \pm \sqrt{b^2 + 4(w-b)^2}}{2(w-b)}$$

$$\cos\theta = \frac{-b + \sqrt{b^2 + 4(w-b)^2}}{2(w-b)} \quad \text{and} \quad \frac{-b - \sqrt{b^2 + 4(w-b)^2}}{2(w-b)}$$

$$\cos\theta = \frac{-b - \sqrt{b^2 + 4(w-b)^2}}{2(w-b)}$$

→ meaningless

Use

$$\cos\theta = \frac{-b + \sqrt{b^2 + 4(w-b)^2}}{2(w-b)}$$

$$x = \tfrac{1}{2}(w-b)\cos\theta$$

$$x = \tfrac{1}{2}(w-b) \times \frac{-b + \sqrt{b^2 + 4(w-b)^2}}{2(w-b)}$$

$$x = \tfrac{1}{4}\left[-b + \sqrt{b^2 + 4(w-b)^2}\right]$$

$a = b + 2x$

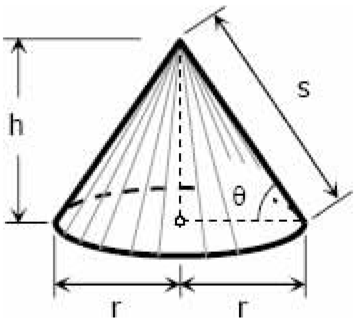
$a = b + 2\left\{ \frac{1}{4} \left[-b + \sqrt{b^2 + 4(w-b)^2} \right] \right\}$

$a = \frac{1}{2}[\,-b + \sqrt{b^2 + 4(w-b)^2}\,]$

answer

Problem 10
Find the largest conical tent that can be constructed having a given slant height.

Solution 10
$V = \frac{1}{3}\pi r^2 h$



Where:
$h = s\sin\theta$

$r = s\cos\theta$

$V = \frac{1}{3}\pi s^3 \cos^2\theta \sin\theta$

$V = \frac{1}{3}\pi s^3 (1 - \sin^2\theta)\sin\theta$

$V = \frac{1}{3}\pi s^3 (\sin\theta - \sin^3\theta)$

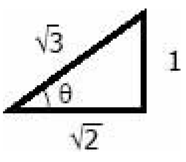
$\frac{dV}{d\theta} = \frac{1}{3}\pi s^3 (\cos\theta - 3\sin^2\theta\cos\theta) = 0$

$\cos\theta - 3\sin^2\theta\cos\theta = 0$

$1 - 3\sin^2\theta = 0$

$\sin^2\theta = 1/3$

$\sin\theta = \sqrt{1/3} = 1/\sqrt{3}$



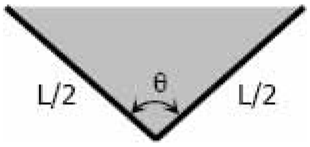
$h = s\sin\theta = \frac{1}{\sqrt{3}}\, s$

$r = s\cos\theta = \sqrt{\frac{2}{3}}\, s$

answer

Problem 11
A gutter having a triangular cross-section is to be made by bending a strip of tin in the middle. Find the angle between the sides when the carrying capacity is to a maximum.

Solution 11



$A = \frac{1}{2}(\frac{1}{2}L)(\frac{1}{2}L)\sin\theta$

$A = \frac{1}{8}L^2\sin\theta$

$\dfrac{dA}{d\theta} = \frac{1}{8}L^2\cos\theta = 0$

$\cos\theta = 0$

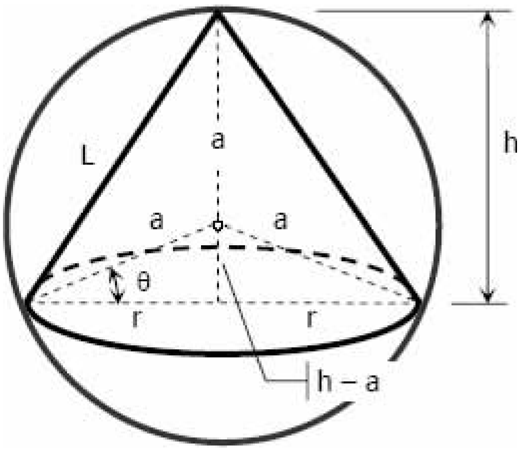
$\theta = 90^\circ$

answer

Problem 12

Find the altitude of the circular cone of maximum convex surface inscribed a sphere of radius a.

Solution 12



$r = a\cos\theta$

$h = a\sin\theta + a = a(\sin\theta + 1)$

$L = \sqrt{h^2 + r^2}$

$L = \sqrt{[\,a(\sin\theta + 1)\,]^2 + (a\cos\theta)^2}$

$L = \sqrt{a^2(\sin\theta + 1)^2 + a^2\cos^2\theta}$

$L = a\sqrt{(\sin\theta + 1)^2 + \cos^2\theta}$

$L = a\sqrt{(\sin^2\theta + 2\sin\theta + 1) + \cos^2\theta}$

$L = a\sqrt{(\sin^2\theta + 2\sin\theta + 1) + \cos^2\theta}$

$L = a\sqrt{2\sin\theta + 1 + (\sin^2\theta + \cos^2\theta)}$

$L = a\sqrt{2\sin\theta + 1 + 1}$

$L = a\sqrt{2\sin\theta + 2}$

Convex area of cone:

$A_L = \pi rL$

$A_L = \pi(a\cos\theta)\left(a\sqrt{2\sin\theta + 2}\right)$

$A_L = \pi a^2\cos\theta\sqrt{2\sin\theta + 2}$

$$\frac{dA_L}{d\theta} = \pi a^2 \left(\cos\theta \frac{2\cos\theta}{2\sqrt{2\sin\theta + 2}} - \sin\theta\sqrt{2\sin\theta + 2} \right) = 0$$

$$\frac{\cos^2\theta}{\sqrt{2\sin\theta + 2}} = \sin\theta\sqrt{2\sin\theta + 2}$$

$$\cos^2\theta = \sin\theta(2\sin\theta + 2)$$

$$1 - \sin^2\theta = 2\sin^2\theta + 2\sin\theta$$

$$3\sin^2\theta + 2\sin\theta - 1 = 0$$

$$(3\sin\theta - 1)(\sin\theta + 1) = 0$$

for
$$3\sin\theta - 1 = 0$$

$$\sin\theta = 1/3$$

for
$$\sin\theta + 1 = 0$$

$$\sin\theta = -1$$
→ (meaningless)

use
$$\sin\theta = 1/3$$

$$h = a(\sin\theta + 1)$$

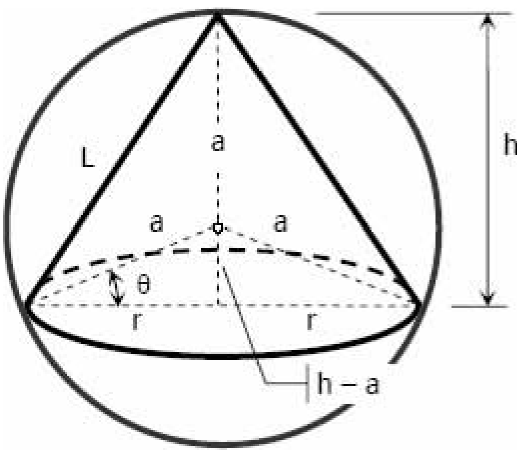
$$h = a(\tfrac{1}{3} + 1)$$

$$h = \tfrac{4}{3}a$$
answer

Problem 13

A sphere is cut in the shape of a circular cone. How much of the material can be saved?

Solution 13



Volume of cone
$V_c = \frac{1}{3}\pi r^2 h$

Where:
$r = a\cos\theta$

$h = a\sin\theta + a = a(\sin\theta + 1)$

$V_c = \frac{1}{3}\pi(a\cos\theta)^2[\,a(\sin\theta+1)\,]$

$V_c = \frac{1}{3}\pi a^3\cos^2\theta(\sin\theta+1)$

$$\frac{dV_c}{d\theta} = \frac{1}{3}\pi a^3\left\{\cos^2\theta(\cos\theta) + (\sin\theta+1)[\,2\cos\theta(-\sin\theta)\,]\right\} = 0$$

$\cos^3\theta - 2\sin\theta\cos\theta(\sin\theta+1) = 0$

$\cos^2\theta - 2\sin\theta(\sin\theta+1) = 0$

$(1-\sin^2\theta) - 2\sin^2\theta - 2\sin\theta = 0$

$1 - 3\sin^2\theta - 2\sin\theta = 0$

$3\sin^2\theta + 2\sin\theta - 1 = 0$

$(3\sin\theta - 1)(\sin\theta + 1) = 0$

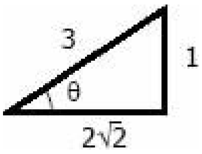
for
$3\sin\theta - 1 = 0$

$\sin\theta = 1/3$

for
$\sin\theta + 1 = 0$

$\sin\theta = -1$
→ (meaningless)

use
$\sin\theta = 1/3$



$$V_c = \frac{1}{3}\pi a^3\left(\frac{2\sqrt{2}}{3}\right)^2\left(\frac{1}{3}+1\right)$$

$V_c = \frac{32}{81}\pi a^3$

Volume of sphere:
$V_s = \frac{4}{3}\pi a^3$

$$\text{Material saved } = \frac{V_c}{V_s} \times 100\%$$

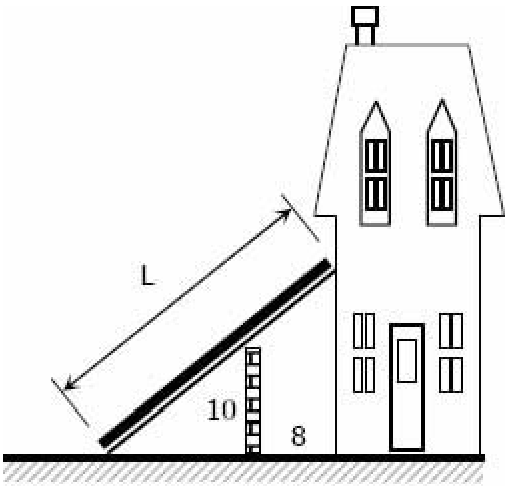
$$\text{Material saved } = \frac{\frac{32}{81}\pi a^3}{\frac{4}{3}\pi a^3} \times 100\%$$

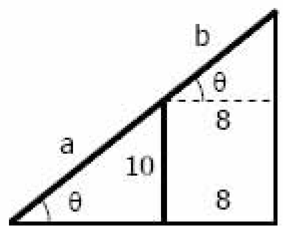
$$\text{Material saved} = 29.63\% \text{ (about } 30\%)$$

answer

Problem 14
A wall 10 ft high is 8 ft from the house. Find the length of the shortest ladder that will reach the house, when one end rests on the ground outside the wall.

Solution 14





$$a = 10 \csc\theta$$

$$b = 8 \sec\theta$$

$$L = a + b$$

$$L = 10 \csc\theta + 8 \sec\theta$$

$$\frac{dL}{d\theta} = -10 \csc\theta \cot\theta + 8 \sec\theta \tan\theta = 0$$

$$8 \sec\theta \tan\theta = 10 \csc\theta \cot\theta$$

$$4\left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right) = 5\left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$\frac{\sin^3\theta}{\cos^3\theta} = \frac{5}{4}$$

$$\tan^3\theta = 5/4$$

$$\tan\theta = \sqrt[3]{5/4}$$

$$\theta = 47.13^\circ$$

$$L = 10 \csc 47.13^\circ + 8 \sec 47.13^\circ$$

$$L = \frac{10}{\sin 47.13^\circ} + \frac{8}{\cos 47.13^\circ}$$

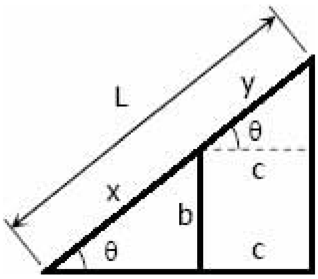
$$L = 25.4 \text{ ft}$$

answer

Problem 15

Solve Problem 14, if the height of the wall is b and its distance from the house is c.

Solution 15



$$x = b\csc\theta$$

$$y = c\sec\theta$$

$$L = x + y$$

$$L = b\csc\theta + c\sec\theta$$

$$\frac{dL}{d\theta} = -b\csc\theta\cot\theta + c\sec\theta\tan\theta = 0$$

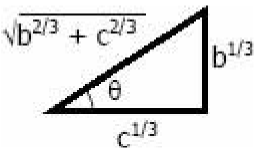
$$c\sec\theta\tan\theta = b\csc\theta\cot\theta$$

$$c\left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right) = b\left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$\frac{\sin^3\theta}{\cos^3\theta} = \frac{b}{c}$$

$$\tan^3\theta = \frac{b}{c}$$

$$\tan\theta = \frac{b^{1/3}}{c^{1/3}}$$



$$L = b\csc\theta + c\sec\theta$$

$$L = b\left(\frac{\sqrt{b^{2/3}+c^{2/3}}}{b^{1/3}}\right) + c\left(\frac{\sqrt{b^{2/3}+c^{2/3}}}{b^{1/3}}\right)$$

$$L = b^{2/3}\sqrt{b^{2/3}+c^{2/3}} + c^{2/3}\sqrt{b^{2/3}+c^{2/3}}$$

$$L = (b^{2/3}+c^{2/3})\sqrt{b^{2/3}+c^{2/3}}$$

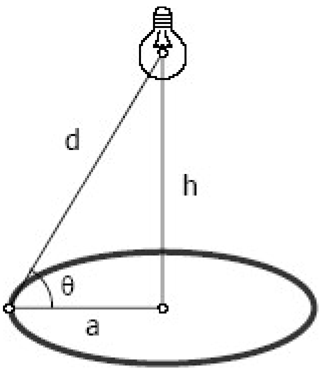
$$L = (b^{2/3}+c^{2/3})(b^{2/3}+c^{2/3})^{1/2}$$

$$L = (b^{2/3}+c^{2/3})^{3/2}$$

answer

Problem 16

A light is to be placed above the center of a circular area of radius a. What height gives the best illumination on a circular walk surrounding the area? (When light from a point source strikes a surface obliquely, the intensity of illumination is



$$I = \frac{k\sin\theta}{d^2}$$

where θ is the angle of incidence and d the distance from the source.)

Solution 16

$$I = \frac{k\sin\theta}{d^2}$$

From the figure:

$$\cos\theta = \frac{a}{d}$$

$$d = \frac{a}{\cos\theta}$$

$$I = \frac{k\sin\theta}{\left(\frac{a}{\cos\theta}\right)^2}$$

$$I = \frac{k\sin\theta}{\frac{a^2}{\cos^2\theta}}$$

$$I = \frac{k}{a^2}\cos^2\theta\ \sin\theta$$

$$I = \frac{k}{a^2}(1-\sin^2\theta)\sin\theta$$

$$I = \frac{k}{a^2}(\sin\theta - \sin^3\theta)$$

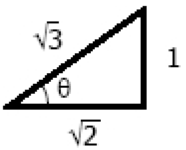
$$\frac{dI}{d\theta} = \frac{k}{a^2}(\cos\theta - 3\sin^2\theta\ \cos\theta) = 0$$

$$\cos\theta - 3\sin^2\theta\ \cos\theta = 0$$

$$1 - 3\sin^2\theta = 0$$

$$\sin^2\theta = 1/3$$

$$\sin\theta = 1/\sqrt{3}$$



$$\tan\theta = \frac{h}{a}$$

$h = a \tan\theta$

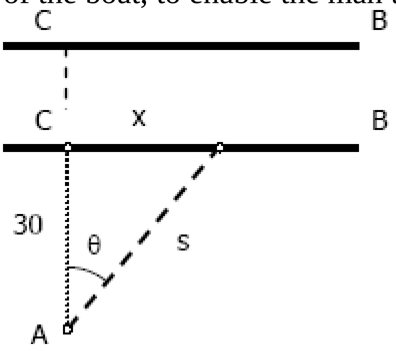
$h = a\left(\frac{1}{\sqrt{2}}\right)$

$h = \frac{1}{\sqrt{2}}\,a = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\,a$

$h = \frac{1}{2}\sqrt{2}\,a$

answer

Problem 17
A man in a motorboat at A receives a message at noon, calling him to B. A bus making 40 miles per hour leaves C, bound for B, at 1:00 PM. If AC = 30 miles, what must be the speed of the boat, to enable the man to catch the bus?



Solution 17

$x = 30\tan\theta$

$s = 30\sec\theta$

$\text{Time}_{\text{boat}} = \text{Time}_{\text{bus+1}}$

Note: time = distance / rate
Let r = rate of boat

Thus,

$$\frac{s}{r} = \frac{x}{40} + 1$$

$$\frac{30\sec\theta}{r} = \frac{30\tan\theta}{40} + 1$$

$$\frac{30\sec\theta}{r} = \frac{3\tan\theta}{4} + 1$$

$$\frac{30\sec\theta}{r} = \frac{3\tan\theta + 4}{4}$$

$$r = \frac{120\sec\theta}{3\tan\theta + 4}$$

$$\frac{dr}{d\theta} = \frac{(3\tan\theta + 4)(120\sec\theta\tan\theta) - 120\sec\theta(3\sec^2\theta)}{(3\tan\theta + 4)^2} = 0$$

$$120\sec\theta\tan\theta(3\tan\theta+4)-360\sec^3\theta=0$$

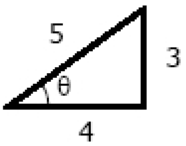
$$\tan\theta(3\tan\theta+4)-3\sec^2\theta=0$$

$$3\tan^2\theta+4\tan\theta-3(1+\tan^2\theta)=0$$

$$3\tan^2\theta+4\tan\theta-3-3\tan^2\theta=0$$

$$4\tan\theta-3=0$$

$$\tan\theta=3/4$$



$$r=\frac{120\sec\theta}{3\tan\theta+4}$$

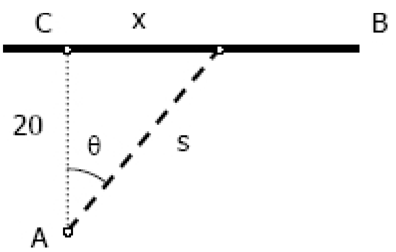
$$r=\frac{120(5/4)}{3(3/4)+4}$$

$$r=24\ \text{miles per hour}$$

answer

Problem 18
Solve Problem 17, if AC = 20 miles and the bus makes 50 miles per hour, leaving C at 12:18 PM, bound for B.

Solution 18



$$x=20\tan\theta$$

$$s=20\sec\theta$$

$$\text{Time}_{\text{boat}}=\text{Time}_{\text{bus}+\frac{18}{60}}$$

Note: time = distance / rate
Let r = rate of boat

Thus,

$$\frac{s}{r}=\frac{x}{50}+\frac{18}{60}$$

$$\frac{20\sec\theta}{r}=\frac{20\tan\theta}{50}+\frac{3}{10}$$

$$1000\sec\theta=20r\tan\theta+15r$$

$$200\sec\theta=(4\tan\theta+3)r$$

$$r = \frac{200 \sec\theta}{4\tan\theta + 3}$$

$$\frac{dr}{d\theta} = \frac{(4\tan\theta + 3)(200\sec\theta \ \tan\theta) - 200\sec\theta(4\sec^2\theta)}{(4\tan\theta+3)^2} = 0$$

$$\frac{200\sec\theta \ \tan\theta\,(4\tan\theta + 3) - 800\sec^3\theta}{(4\tan\theta+3)^2} = 0$$

$$200\sec\theta \ \tan\theta\,(4\tan\theta + 3) - 800\sec^3\theta = 0$$

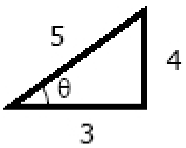
$$\tan\theta\,(4\tan\theta + 3) - 4\sec^2\theta = 0$$

$$4\tan^2\theta + 3\tan\theta - 4(1 + \tan^2\theta) = 0$$

$$4\tan^2\theta + 3\tan\theta - 4 - 4\tan^2\theta = 0$$

$$3\tan\theta - 4 = 0$$

$$\tan\theta = 4/3$$



$$r = \frac{200 \sec\theta}{4\tan\theta + 3}$$

$$r = \frac{200(5/3)}{4(4/3) + 3}$$

$$r = 40 \ \text{miles per hour}$$

answer

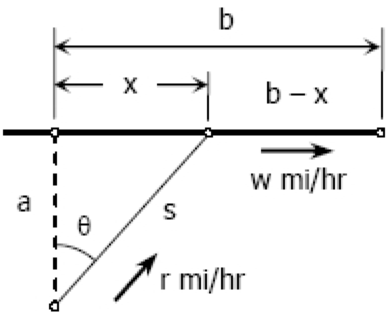
Problem 19

A man on an island *a* miles south of a straight beach wishes to reach a point on shore *b* miles east of his present position. If he can row *r* miles per hour and walk *w* miles per hour, in what direction should he row, to reach his destination as soon as possible? See Fig. 57.

Solution 19

Time to row:

$$t_1 = \frac{s}{r}$$



Time to walk:

$$t_2 = \frac{b - x}{w}$$

Total time:

$$t = t_1 + t_2$$

$$t = \frac{s}{r} + \frac{b - x}{w}$$

From the figure:

$$x = a \tan\theta$$

$$s = a \sec\theta$$

$$t = \frac{a \sec\theta}{r} + \frac{b - a \tan\theta}{w}$$

$$\frac{dt}{d\theta} = \frac{a \sec\theta \, \tan\theta}{r} - \frac{a \sec^2\theta}{w} = 0$$

$$\frac{a \sec\theta \, \tan\theta}{r} - \frac{a \sec^2\theta}{w} = 0$$

$$\frac{\tan\theta}{r} - \frac{\sec\theta}{w} = 0$$

$$\frac{\sin\theta}{r \, \cos\theta} - \frac{1}{w \, \cos\theta} = 0$$

$$\frac{\sin\theta}{r \, \cos\theta} = \frac{1}{w \, \cos\theta}$$

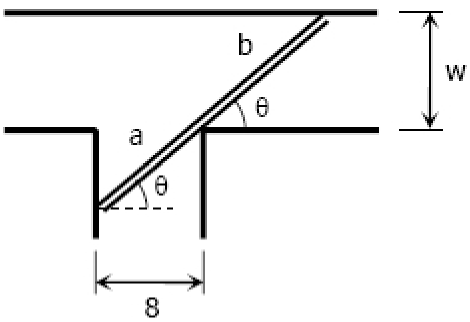
$$\sin\theta = \frac{r}{w}$$

answer

Problem 20

A pole 27 feet long is carried horizontally along a corridor 8 feet wide and into a second corridor at right angles to the first. How wide must the second corridor be?

Solution 20



$$a = 8 \sec\theta$$

$$b = w \csc\theta$$

$$a + b = 24$$

$$8 \sec\theta + w \csc\theta = 24$$

$$w = \frac{24 - 8 \sec\theta}{\csc\theta}$$

$$w = 24 \frac{1}{\csc\theta} - 8 \frac{\sec\theta}{\csc\theta}$$

$w = 24 \sin \theta - 8 \dfrac{\sin \theta}{\cos \theta}$

$w = 24 \sin \theta - 8 \tan \theta$

$\dfrac{dw}{\theta} = 24 \cos \theta - 8 \sec^2 \theta = 0$

$24 \cos \theta = 8 \sec^2 \theta$

$24 \cos \theta = \dfrac{8}{\cos^2 \theta}$

$\cos^3 \theta = \dfrac{8}{24}$

$\cos^3 \theta = \dfrac{1}{3}$

$\theta = 46.10°$

$w = 24 \sin \theta - 8 \tan \theta$

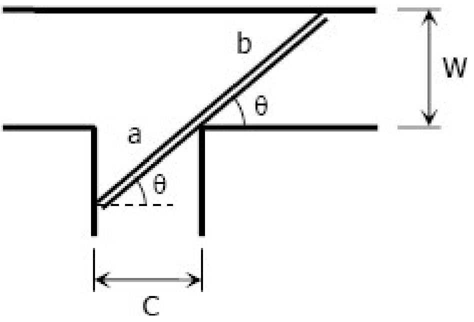
$w = 24 \sin 46.10° - 8 \tan 46.10°$

$w = 8.98 \text{ ft}$ *answer*

Problem 21

Solve Problem 20 if the pole is of length L and the first corridor is of width C.

Solution 21



$a = C \sec \theta$

$b = W \csc \theta$

$a + b = L$

$C \sec \theta + W \csc \theta = L$

$W = \dfrac{L - C \sec \theta}{\csc \theta}$

$W = \dfrac{L}{\csc \theta} - \dfrac{C \sec \theta}{\csc \theta}$

$W = L \sin \theta - \dfrac{C \sin \theta}{\cos \theta}$

$W = L \sin \theta - C \tan \theta$

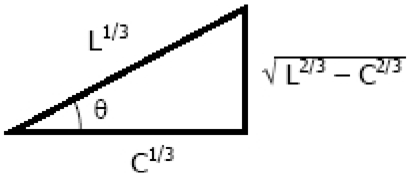
$$\frac{dW}{d\theta} = L\cos\theta - C\sec^2\theta$$

$$L\cos\theta = C\sec^2\theta$$

$$L\cos\theta = \frac{C}{\cos^2\theta}$$

$$\cos^3\theta = \frac{C}{L}$$

$$\cos\theta = \frac{C^{1/3}}{L^{1/3}}$$



$$W = L\sin\theta - C\tan\theta$$

$$W = L\left(\frac{\sqrt{L^{2/3} - C^{2/3}}}{L^{1/3}}\right) - C\left(\frac{\sqrt{L^{2/3} - C^{2/3}}}{C^{1/3}}\right)$$

$$W = L^{2/3}\sqrt{L^{2/3} - C^{2/3}} - C^{2/3}\sqrt{L^{2/3} - C^{2/3}}$$

$$W = \left(L^{2/3} - C^{2/3}\right)\sqrt{L^{2/3} - C^{2/3}}$$

$$W = \left(L^{2/3} - C^{2/3}\right)\left(L^{2/3} - C^{2/3}\right)^{1/2}$$

$$W = \left(L^{2/3} - C^{2/3}\right)^{3/2}$$

answer

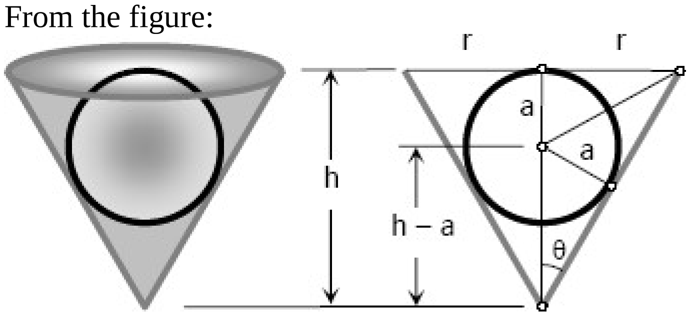
(You may check the answer of Problem 20 above by using this formula)

Problem 22
A sphere of radius *a* is dropped into a conical vessel full of water. Find the altitude of the smallest cone that will permit the sphere to be entirely submerged.

Solution 22

Volume of cone:
$V = \frac{1}{3}\pi r^2 h$

From the figure:



$$\csc\theta = \frac{h - a}{a}$$

$$h = a\csc\theta + a$$

$$h = a(\csc\theta + 1)$$

$$\tan\theta = \frac{r}{h}$$

$$r = h\tan\theta$$

$$r = a(\csc\theta + 1)\tan\theta$$

$$V = \tfrac{1}{3}\pi[\,a(\csc\theta + 1)\tan\theta\,]^2[\,a(\csc\theta + 1)\,]$$

$$V = \tfrac{1}{3}\pi a^3(\csc\theta + 1)^3\tan^2\theta$$

$$\frac{dV}{d\theta} = \tfrac{1}{3}\pi a^3\left[(\csc\theta + 1)^3(2\tan\theta\sec^2\theta) + 3\tan^2\theta(\csc\theta + 1)^2(-\csc\theta\cot\theta)\right] = 0$$

$$2\tan\theta\sec^2\theta(\csc\theta + 1)^3 - 3\tan^2\theta\csc\theta\cot\theta(\csc\theta + 1)^2 = 0$$

$$2\sec^2\theta(\csc\theta + 1) - 3\tan\theta\csc\theta\cot\theta = 0$$

$$2\left(\frac{1}{\cos^2\theta}\right)\left(\frac{1}{\sin\theta} + 1\right) - 3\tan\theta\left(\frac{1}{\sin\theta}\right)\left(\frac{1}{\tan\theta}\right) = 0$$

$$2\left(\frac{1}{\cos^2\theta}\right)\left(\frac{1}{\sin\theta} + 1\right) = 3\left(\frac{1}{\sin\theta}\right)$$

$$2\sin\theta\left(\frac{1}{\sin\theta} + 1\right) = 3\cos^2\theta$$

$$2 + 2\sin\theta = 3\cos^2\theta$$

$$2 + 2\sin\theta = 3(1 - \sin^2\theta)$$

$$3\sin^2\theta + 2\sin\theta - 1 = 0$$

$$(3\sin\theta - 1)(\sin\theta + 1) = 0$$

for
$$3\sin\theta - 1 = 0$$

$$\sin\theta = 1/3$$

for
$$\sin\theta + 1 = 0$$

$$\sin\theta = -1$$
→ (meaningless)

use
$\sin\theta = 1/3$

$$h = a(\csc\theta + 1)$$

$$h = a\left(\frac{1}{\sin\theta} + 1\right)$$

$$h = a\left(\frac{1}{1/3} + 1\right)$$

$h = 4a$ *answer*

Problem 24
Find the area of the largest rectangle that can be cut from a circular quadrant as in Fig. 76.

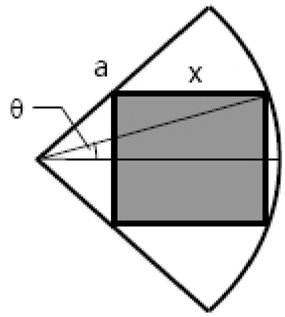
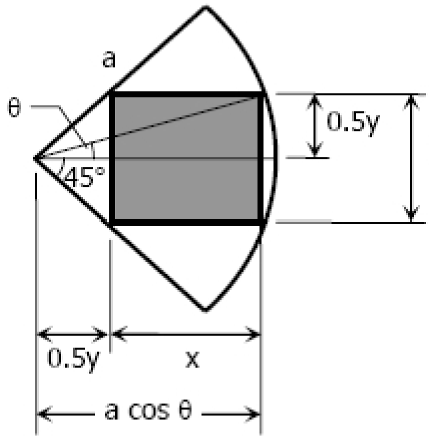


Figure 76



Solution 24
From the figure:
$0.5y = a\sin\theta$

$$y = 2a\sin\theta$$

$$x + 0.5y = a\cos\theta$$

$$x = a\cos\theta - 0.5y$$

$$x = a\cos\theta - 0.5(2a\sin\theta)$$

$$x = a(\cos\theta - \sin\theta)$$

Area of the rectangle:
$A = xy$

$$A = [\,a(\cos\theta - \sin\theta)\,](2a\sin\theta)$$

$$A = 2a^2 \sin\theta(\cos\theta - \sin\theta)$$

$$\frac{dA}{d\theta} = 2a^2[\,\sin\theta(-\sin\theta - \cos\theta) + (\cos\theta - \sin\theta)\cos\theta\,] = 0$$

$$-\sin\theta(\sin\theta + \cos\theta) + \cos\theta(\cos\theta - \sin\theta) = 0$$

$$-\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta - \sin\theta\cos\theta = 0$$

$$(\cos^2\theta - \sin^2\theta) - 2\sin\theta\cos\theta = 0$$

$$\cos 2\theta - \sin 2\theta = 0$$

$$\sin 2\theta = \cos 2\theta$$

$$\frac{\sin 2\theta}{\cos 2\theta} = 1$$

$$\tan 2\theta = 1$$

$$\theta = 22.5^\circ$$

$$A = 2a^2 \sin 22.5^\circ(\cos 22.5^\circ - \sin 22.5^\circ)$$

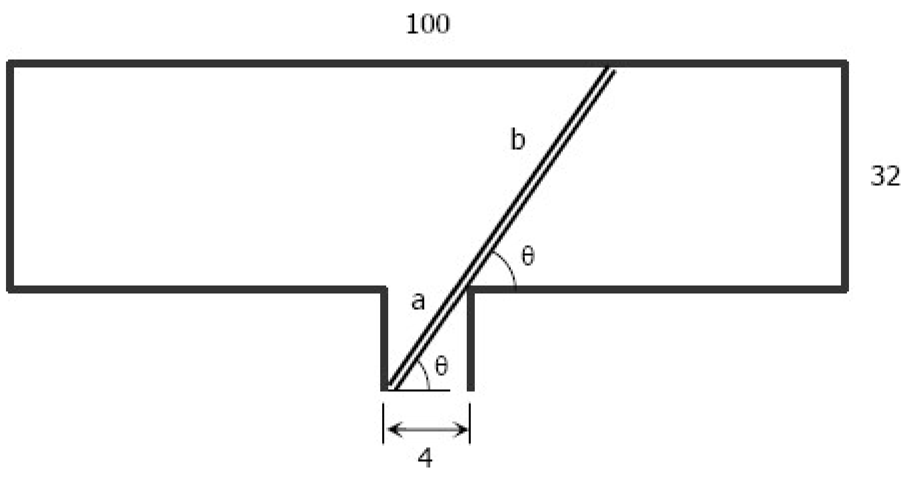
$$A = 0.4142a^2$$

answer

Problem 26

A corridor 4 ft wide opens into a room 100 ft long and 32 ft wide, at the middle of one side. Find the length of the longest thin rod that can be carried horizontally into the room.

Solution 26



$$a = 4\sec\theta$$

$$b = 32\csc\theta$$

Total length of rod:

$$L = a + b$$

$$L = 4\sec\theta + 32\csc\theta$$

$$\frac{dL}{d\theta} = 4\sec\theta\tan\theta - 32\csc\theta\cot\theta = 0$$

$$\sec\theta\tan\theta - 8\csc\theta\cot\theta = 0$$

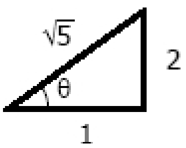
$$\sec\theta\tan\theta = 8\csc\theta\cot\theta$$

$$\frac{1}{\cos\theta}\left(\frac{\sin\theta}{\cos\theta}\right) = 8\left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$\frac{\sin^3\theta}{\cos^3\theta} = 8$$

$$\tan^3\theta = 8$$

$$\tan\theta = 2$$



$$L = 4\sec\theta + 32\csc\theta$$

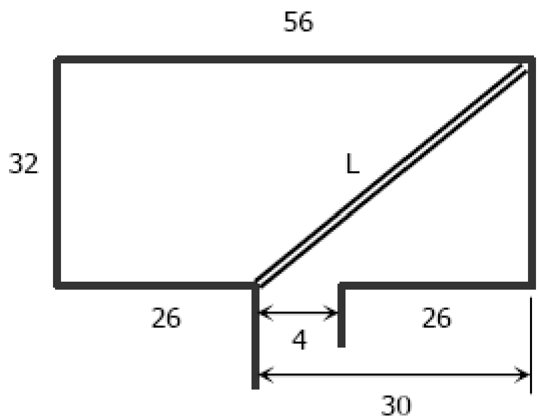
$$L = 4\left(\frac{\sqrt{5}}{1}\right) + 32\left(\frac{\sqrt{5}}{2}\right)$$

$$L = 20\sqrt{5} = 44.72\ \text{ ft}$$

answer

Problem 27
Solve Problem 26 if the room is 56 feet long.

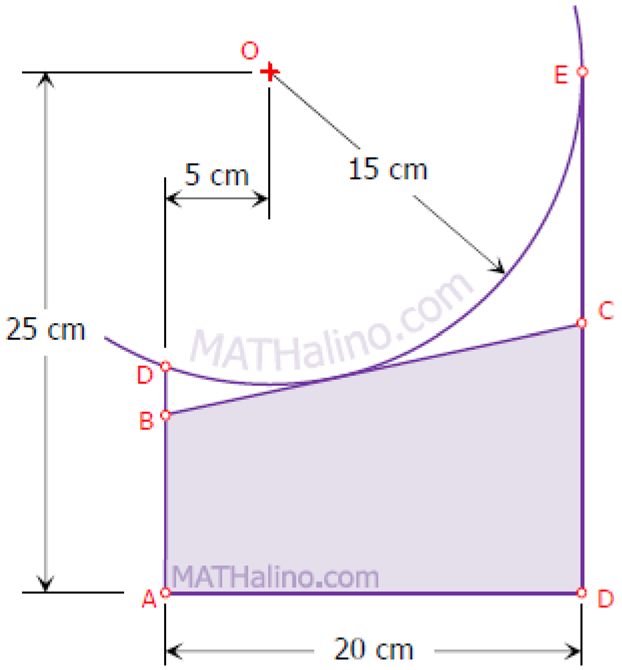
Solution 27



From the figure:
$L = \sqrt{30^2 + 32^2}$

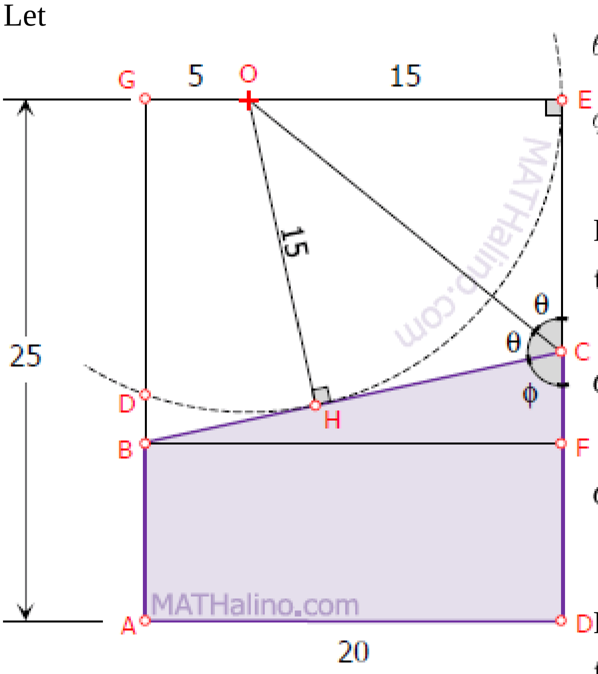
$L = 43.86\ \text{ ft}$
answer

BC of trapezoid ABCD is tangent at any point on circular arc DE whose center is O. Find the length of BC so that the area of ABCD is maximum.



Solution

Let



θ = ∠HCO = ∠ECO

$\phi = 180° - 2\theta$

From right triangle BFC

$\tan\phi = \dfrac{20}{CF}$

$CF = \dfrac{20}{\tan\phi} = \dfrac{20}{\tan(180° - 2\theta)}$

$CF = -\dfrac{20}{\tan 2\theta}$

From right triangle CEO

$\tan\theta = \dfrac{15}{CE}$

$$CE = \frac{15}{\tan\theta}$$

$$BG = CE + CF$$

$$BG = \frac{15}{\tan\theta} - \frac{20}{\tan 2\theta}$$

Area of trapezoid BCEG

$$A_{BCEG} = \tfrac{1}{2}(CE + BG)(20)$$

$$A_{BCEG} = 10\left[\frac{15}{\tan\theta} + \left(\frac{15}{\tan\theta} - \frac{20}{\tan 2\theta}\right)\right]$$

$$A_{BCEG} = \frac{300}{\tan\theta} - \frac{200}{\tan 2\theta}$$

Area of trapezoid ABCD

$$A = A_{ADEG} - A_{BCEG}$$

$$A = 25(20) - \left(\frac{300}{\tan\theta} - \frac{200}{\tan 2\theta}\right)$$

$$A = 500 - 300\cot\theta + 200\cot 2\theta$$

For maximum area, dA/dθ = 0:

$$\frac{dA}{d\theta} = -300(-\csc^2\theta) + 200(-2\csc^2 2\theta) = 0$$

$$300\csc^2\theta - 400\csc^2 2\theta = 0$$

$$3\csc^2\theta = 4\csc^2 2\theta$$

$$\sqrt{3}\csc\theta = 2\csc 2\theta$$

$$\frac{\sqrt{3}}{\sin\theta} = \frac{2}{\sin 2\theta}$$

$$\frac{\sqrt{3}}{\sin\theta} = \frac{2}{2\sin\theta\cos\theta}$$

$$\cos\theta = \frac{1}{\sqrt{3}}$$

$$\theta = 54.74^\circ$$

$$\phi = 180^\circ - 2\theta = 180^\circ - 2(54.74^\circ)$$

$$\phi = 70.53^\circ$$

From right triangle BFC

$$\sin\phi = \frac{20}{BC}$$

$$BC = \frac{20}{\sin\phi} = \frac{20}{\sin 70.53^\circ}$$

$$BC = 21.21 \text{ cm}$$

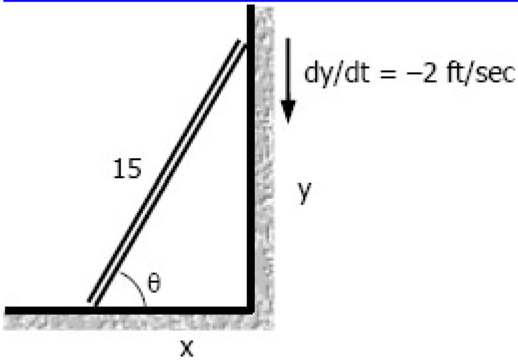
answer

Problems in Caculus Involving Inverse Trigonometric Functions

Problem 37
A ladder 15 ft long leans against a vertical wall. If the top slides down at 2 ft/sec, how fast is the angle of elevation of the ladder decreasing, when the lower end is 12 ft from the wall?

Solution 37

Hide Click here to show or hide the solution



$$\theta = \arcsin \frac{y}{15}$$

$$\frac{d\theta}{dt} = \frac{\dfrac{dy/dt}{15}}{\sqrt{1 - \left(\dfrac{y}{15}\right)^2}}$$

$$\frac{d\theta}{dt} = \frac{\dfrac{dy/dt}{15}}{\sqrt{1 - \left(\dfrac{y}{15}\right)^2}}$$

$$\frac{d\theta}{dt} = \frac{dy/dt}{15\sqrt{1 - \dfrac{y^2}{225}}}$$

when $x = 12$

$y = \sqrt{15^2 - 12^2}$

$y = 9 \text{ ft}$

$$\frac{d\theta}{dt} = \frac{-2}{15\sqrt{1 - \dfrac{9^2}{225}}}$$

$$\frac{d\theta}{dt} = \frac{-2}{15\sqrt{1 - \dfrac{81}{225}}}$$

$$\frac{d\theta}{dt} = \frac{-2}{15\sqrt{1 - \dfrac{9}{25}}}$$

$$\frac{d\theta}{dt} = \frac{-2}{15\sqrt{\frac{16}{25}}}$$

$$\frac{d\theta}{dt} = \frac{-2}{15\left(\frac{4}{5}\right)}$$

$$\frac{d\theta}{dt} = \frac{-2}{12}$$

$$\frac{d\theta}{dt} = -\frac{1}{6}\ \text{rad/sec}$$

answer

The negative sign indicates that the angle is decreasing.

Another Solution

(*Using Trigonometric Function*)

$\sin\theta = y/15$

$$\cos\theta\frac{d\theta}{dt} = \frac{dy/dt}{15}$$

when $x = 12$

$\cos\theta = 12/15$

Thus,

$$\frac{12}{15}\frac{d\theta}{dt} = \frac{-2}{15}$$

$$\frac{d\theta}{dt} = \frac{-2}{15}\times\frac{15}{12}$$

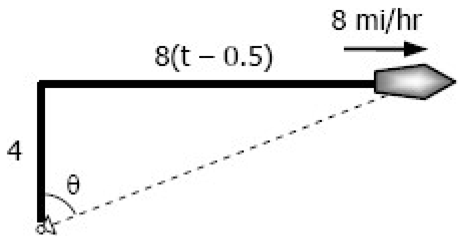
$$\frac{d\theta}{dt} = -\frac{1}{6}\ \text{rad/sec}$$

(*okay!*)

Problem 38

A ship, moving 8 mi/hr, sails north for 30 min, then turns east. If a searchlight at the point of departure follows the ship, how fast is the light rotating 2 hr after the start.

Solution 38



$$\theta = \arctan\frac{8(t-0.5)}{4}$$

$\theta = \arctan 2(t - 0.5)$

$$\frac{d\theta}{dt} = \frac{2}{1 + 4(t - 0.5)^2}$$

after t = 2 hrs

$$\frac{d\theta}{dt} = \frac{2}{1 + 4(2 - 0.5)^2}$$

$$\frac{d\theta}{dt} = \frac{2}{1 + 4(2.25)}$$

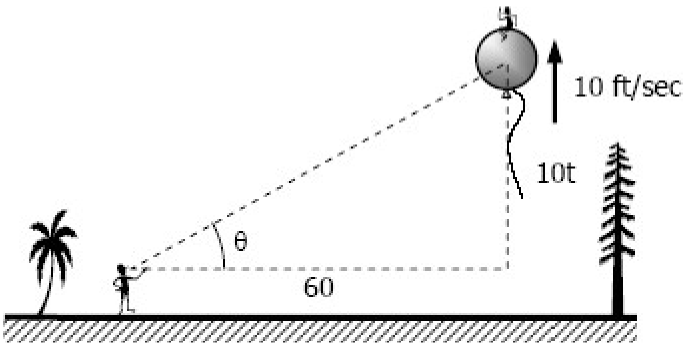
$$\frac{d\theta}{dt} = 0.2 \text{ rad/hr}$$

answer

Problem 39

A balloon, leaving the ground 60 ft from an observer, rises 10 ft/sec. How fast is the angle of elevation of the line of sight increasing, after 8 seconds?

Solution 39



$\theta = \arctan(10t/60)$

$\theta = \arctan \frac{1}{6}t$

$$\frac{d\theta}{dt} = \frac{\frac{1}{6}}{1 + \frac{1}{36}t^2}$$

after t = 8 sec

$$\frac{d\theta}{dt} = \frac{\frac{1}{6}}{1 + \frac{1}{36}(8^2)}$$

$$\frac{d\theta}{dt} = \frac{\frac{1}{6}}{1 + \frac{64}{36}}$$

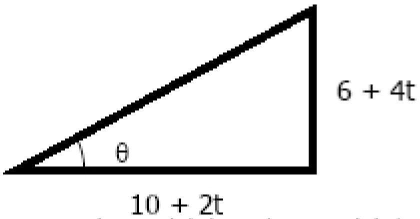
$$\frac{d\theta}{dt} = 0.06 \text{ rad/sec}$$

answer

Problem 40

The base of a right triangle grows 2 ft/sec, the altitude grows 4 ft/sec. If the base and altitude are originally 10 ft and 6 ft, respectively, find the time rate of change of the base angle, when the angle is 45°.

Solution 40



$$\theta = \arctan\frac{6+4t}{10+2t}$$

$$\theta = \arctan\frac{3+2t}{5+t}$$

$$\frac{d\theta}{dt} = \frac{\dfrac{(5+t)(2)-(3+2t)(1)}{(5+t)^2}}{1+\left(\dfrac{3+2t}{5+t}\right)^2}$$

$$\frac{d\theta}{dt} = \frac{\dfrac{10+2t-3-2t}{(5+t)^2}}{\dfrac{(5+t)^2+(3+2t)^2}{(5+t)^2}}$$

$$\frac{d\theta}{dt} = \frac{7}{(5+t)^2+(3+2t)^2}$$

when $\theta = 45^\circ$

$$6+4t = 10+2t$$

$$2t = 4$$

$$t = 2\ \text{sec}$$

Thus,

$$\frac{d\theta}{dt} = \frac{7}{(5+2)^2+[\,3+2(2)\,]^2}$$

$$\frac{d\theta}{dt} = \frac{7}{49+49}$$

$$\frac{d\theta}{dt} = \frac{7}{98}$$

$$\frac{d\theta}{dt} = \frac{1}{14}\ \text{rad/sec}$$

answer

Another Solution

$$\tan\theta = \frac{6+4t}{10+2t}$$

$$\tan\theta = \frac{3+2t}{5+t}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{(5+t)(2)-(3+2t)(1)}{(5+t)^2}$$

$$\frac{1}{\cos^2\theta} \times \frac{d\theta}{dt} = \frac{10+2t-3-2t}{(5+t)^2}$$

$$\frac{d\theta}{dt} = \cos^2\theta \times \frac{7}{(5+t)^2}$$

when $\theta = 45°$

$$6+4t = 10+2t$$

$$2t = 4$$

$$t = 2 \text{ sec}$$

Thus,

$$\frac{d\theta}{dt} = \cos^2 45° \times \frac{7}{(5+2)^2}$$

$$\frac{d\theta}{dt} = \left(\frac{1}{\sqrt{2}}\right)^2 \left(\frac{7}{7^2}\right)$$

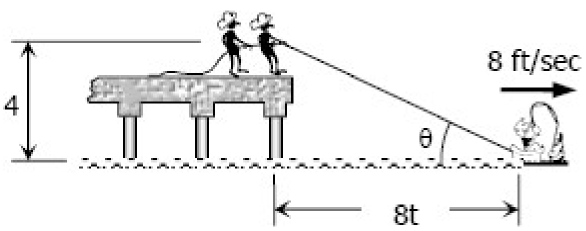
$$\frac{d\theta}{dt} = \frac{1}{2} \times \frac{1}{7}$$

$$\frac{d\theta}{dt} = \frac{1}{14} \text{ rad/sec}$$

(*okay!*)

Problem 44

A rowboat is pushed off from a beach at 8 ft/sec. A man on shore holds a rope, tied to the boat, at a height of 4 ft. Find how fast the angle of elevation of the rope is decreasing, after 1 sec.



Solution 44

$\theta = \arctan(4/8t)$

$$\theta = \arctan\frac{1}{2t}$$

$$\frac{d\theta}{dt} = \frac{\frac{-1(2)}{(2t)^2}}{1+\left(\frac{1}{2t}\right)^2}$$

$$\frac{d\theta}{dt} = \frac{\frac{-1}{2t^2}}{1 + \frac{1}{4t^2}}$$

$$\frac{d\theta}{dt} = -\frac{1}{2t^2\left(\frac{4t^2+1}{4t^2}\right)}$$

$$\frac{d\theta}{dt} = -\frac{1}{\frac{4t^2+1}{2}}$$

$$\frac{d\theta}{dt} = -\frac{2}{4t^2+1}$$

after t = 1 sec

$$\frac{d\theta}{dt} = -\frac{2}{4(1^2)+1}$$

$$\frac{d\theta}{dt} = -\frac{2}{5} \text{ rad/sec}$$

answer

The negative sign indicates that the angle is decreasing.

Another Solution

$$\tan\theta = 4/8t$$

$$\tan\theta = \frac{1}{2t}$$

$$\sec^2\theta \, \frac{d\theta}{dt} = \frac{-1(2)}{(2t)^2}$$

$$\frac{1}{\cos^2\theta} \, \frac{d\theta}{dt} = -\frac{1}{2t^2}$$

$$\frac{d\theta}{dt} = -\cos^2\theta \, \frac{1}{2t^2}$$

after t = 1 sec

$$8t = 8(1) = 8$$

$$\text{hypotenuse} = \sqrt{8^3 + 4^2} = 4\sqrt{5}$$

$$\cos\theta = \frac{8}{4\sqrt{5}}$$

$$\frac{d\theta}{dt} = -\left(\frac{8}{4\sqrt{5}}\right)^2 \frac{1}{2(1^2)}$$

$$\frac{d\theta}{dt} = -\left(\frac{64}{80}\right)\frac{1}{2}$$

$$\frac{d\theta}{dt} = -\left(\frac{64}{80}\right)\frac{1}{2}$$

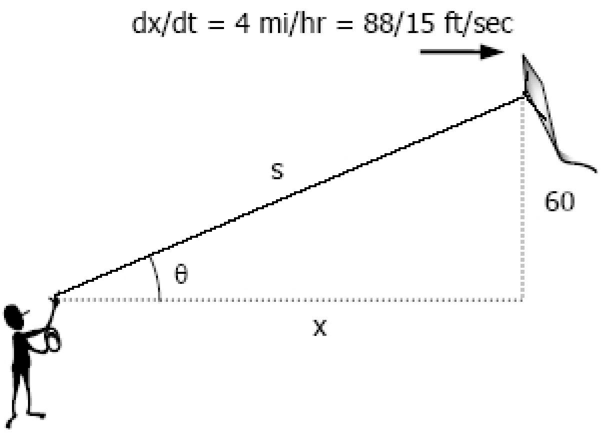
$$\frac{d\theta}{dt} = -\frac{2}{5} \text{ rad/sec}$$

(*okay!*)

Problem 45

A kite is 60 ft high with 100 ft of cord out. If the kite is moving horizontally 4 mi/hr directly away from the boy flying it, find the rate of change of the angle of elevation of the cord.

Solution 45



$$\theta = \arctan\frac{60}{x}$$

$$\frac{d\theta}{dt} = \frac{\dfrac{-60\frac{dx}{dt}}{x^2}}{1+\left(\dfrac{60}{x}\right)^2}$$

$$\frac{d\theta}{dt} = \frac{-60\frac{dx}{dt}}{x^2\left(1+\dfrac{3600}{x^2}\right)}$$

$$\frac{d\theta}{dt} = \frac{-60\frac{dx}{dt}}{x^2\left(1+\dfrac{3600}{x^2}\right)}$$

$$\frac{d\theta}{dt} = \frac{-60\frac{dx}{dt}}{x^2+3600}$$

when s = 100 ft

$x = \sqrt{100^2 - 60^2}$

$x = 80 \text{ ft}$

$$\frac{d\theta}{dt} = \frac{-60(88/15)}{80^2 + 3600}$$

$$\frac{d\theta}{dt} = \frac{-352}{10\,000}$$

$$\frac{d\theta}{dt} = -\frac{22}{625} \text{ rad/sec}$$

$$\frac{d\theta}{dt} = \frac{22}{625} \text{ rad/sec decreasing}$$

answer

Another Solution

$$\tan\theta = \frac{60}{x}$$

$$\sec^2\theta \, \frac{d\theta}{dt} = \frac{-60\dfrac{dx}{dt}}{x^2}$$

$$\frac{1}{\cos^2\theta} \, \frac{d\theta}{dt} = -\frac{60\dfrac{dx}{dt}}{x^2}$$

$$\frac{d\theta}{dt} = -\frac{60\cos^2\theta}{x^2} \, \frac{dx}{dt}$$

When s = 100 ft
x = 80 ft (see Solution 45 above)
cos θ = 80/100 = 4/5

$$\frac{d\theta}{dt} = -\frac{60(4/5)^2}{80^2} \times (88/15)$$

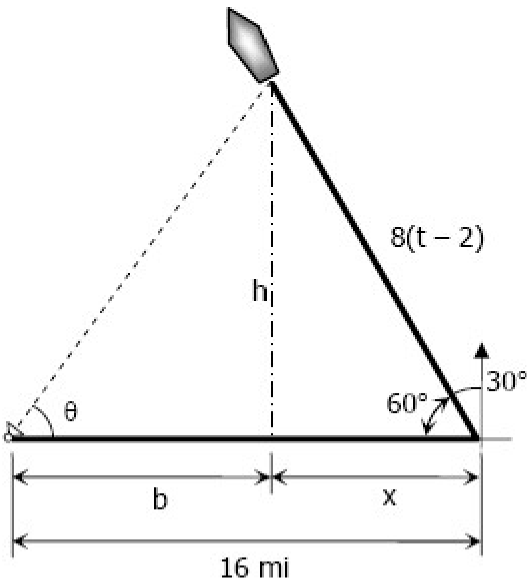
$$\frac{d\theta}{dt} = -\frac{22}{625} \text{ rad/sec}$$

(*okay!*)

Problem 46

A ship, moving at 8 mi/hr, sails east for 2 hr, then turns N 30° W. A searchlight, placed at the starting point, follows the ship. Find how fast the light is rotating, (a) 3 hr after the start; (b) just after the turn.

Solution 46



$$h = 8(t-2)\sin 60^\circ$$

$$h = 8(t-2)\left(\frac{\sqrt{3}}{2}\right)$$

$$h = 4\sqrt{3}(t-2)$$

$$x = 8(t-2)\cos 60^\circ$$

$$x = 8(t-2)(1/2)$$

$$x = 4(t-2)$$

$$b = 16 - x = 16 - 4(t-2)$$

$$b = 16 - 4t + 8 = 24 - 4t$$

$$b = 4(6-t)$$

$$\theta = \arctan\frac{h}{b} = \arctan\frac{4\sqrt{3}(t-2)}{4(6-t)}$$

$$\theta = \arctan\frac{\sqrt{3}(t-2)}{6-t}$$

$$\frac{d\theta}{dt} = \frac{\dfrac{(6-t)\sqrt{3} - \sqrt{3}(t-2)(-1)}{(6-t)^2}}{1 + \left[\dfrac{\sqrt{3}(t-2)}{6-t}\right]^2}$$

$$\frac{d\theta}{dt} = \frac{\dfrac{6\sqrt{3} - t\sqrt{3} + t\sqrt{3} - 2\sqrt{3}}{(6-t)^2}}{1 + \dfrac{3(t-2)^2}{(6-t)^2}}$$

$$\frac{d\theta}{dt} = \frac{\dfrac{4\sqrt{3}}{(6-t)^2}}{\dfrac{(6-t)^2 + 3(t-2)^2}{(6-t)^2}}$$

$$\frac{d\theta}{dt} = \frac{4\sqrt{3}}{(6-t)^2 + 3(t-2)^2}$$

(a) 3 hours after the start, t = 3

$$\dfrac{d\theta}{dt} = \dfrac{4\sqrt{3}}{(6-3)^2 + 3(3-2)^2}$$

$$\dfrac{d\theta}{dt} = \dfrac{4\sqrt{3}}{9+3}$$

$$\dfrac{d\theta}{dt} = \dfrac{4\sqrt{3}}{12}$$

$$\dfrac{d\theta}{dt} = \tfrac{1}{3}\sqrt{3} \ \text{rad/sec}$$

answer

(b) Just after the turn, t = 2

$$\dfrac{d\theta}{dt} = \dfrac{4\sqrt{3}}{(6-2)^2 + 3(2-2)^2}$$

$$\dfrac{d\theta}{dt} = \dfrac{4\sqrt{3}}{16+0}$$

$$\dfrac{d\theta}{dt} = \dfrac{4\sqrt{3}}{16}$$

$$\dfrac{d\theta}{dt} = \tfrac{1}{4}\sqrt{3} \ \text{rad/sec}$$

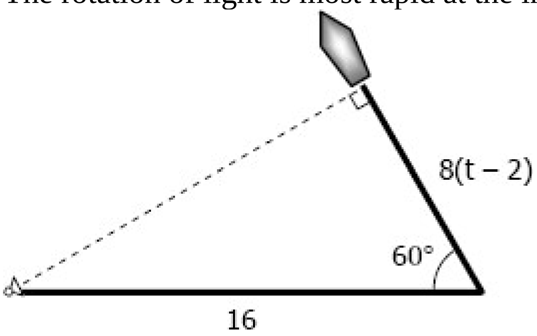
answer

Problem 47

In Problem 46, find when the light rotates most rapidly.

Solution 47

The rotation of light is most rapid at the instant when the light is perpendicular to the path of the ship.



$$\cos 60^\circ = \dfrac{8(t-2)}{16}$$

$$16\cos 60^\circ = 8(t-2)$$

$$8 = 8(t-2)$$

$$1 = t - 2$$

$$t = 3 \ \text{hrs}$$

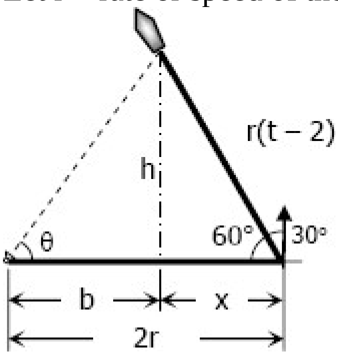
answer

Problem 48

Prove that the result in Problem 46 and Problem 47 are independent of the speed of the ship.

Solution 48

Let r = rate or speed of the ship



$$h = r(t-2)\sin 60°$$

$$h = \tfrac{1}{2}\sqrt{3}r(t-2)$$

$$x = r(t-2)\cos 60°$$

$$x = \tfrac{1}{2}r(t-2)$$

$$b = 2r - x = 2r - \tfrac{1}{2}r(t-2)$$

$$b = 2r - \tfrac{1}{2}rt + r = 3r - \tfrac{1}{2}rt$$

$$b = \tfrac{1}{2}r(6-t)$$

$$\theta = \arctan\frac{h}{b} = \arctan\frac{\tfrac{1}{2}\sqrt{3}r(t-2)}{\tfrac{1}{2}r(6-t)}$$

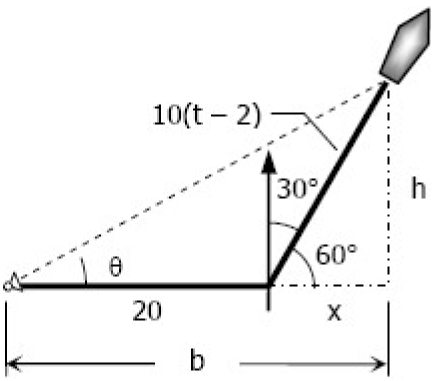
$$\theta = \arctan\frac{h}{b} = \arctan\frac{\sqrt{3}(t-2)}{6-t}$$

The above value of angle θ did not include the speed of the ship r. The derivative of θ in terms of time t will of course be independent from r. Solutions 46 & 47 are both derivative of θ in terms of time t.

Problem 49

A ship, moving 10 mi/hr, sails east for 2 hours, then turns N 30° E. A searchlight, placed at the starting point, follows the ship. Find how fast the light is rotating (a) 4 hours after the start; (b) just after the turn.

Solution 49



$$h = 10(t-2)\sin 60°$$

$$h = 10(t-2)\left(\frac{\sqrt{3}}{2}\right)$$

$$h = 5\sqrt{3}(t-2)$$

$$x = 10(t-2)\cos 60°$$

$$x = 10(t-2)(1/2)$$

$$x = 5(t-2)$$

$$b = 20 + x = 20 + 5(t-2)$$

$$b = 20 + 5t - 10 = 5t + 10$$

$$b = 5(t+2)$$

$$\theta = \arctan \frac{h}{b}$$

$$\theta = \arctan \frac{5\sqrt{3}(t-2)}{5(t+2)}$$

$$\theta = \arctan \frac{\sqrt{3}(t-2)}{t+2}$$

$$\frac{d\theta}{dt} = \frac{\dfrac{(t+2)\sqrt{3} - \sqrt{3}(t-2)(1)}{(t+2)^2}}{1 + \left[\dfrac{\sqrt{3}(t-2)}{t+2}\right]^2}$$

$$\frac{d\theta}{dt} = \frac{\dfrac{t\sqrt{3} + 2\sqrt{3} - t\sqrt{3} + 2\sqrt{3}}{(t+2)^2}}{1 + \dfrac{3(t-2)^2}{(t+2)^2}}$$

$$\frac{d\theta}{dt} = \frac{\dfrac{4\sqrt{3}}{(t+2)^2}}{\dfrac{(t+2)^2 + 3(t-2)^2}{(t+2)^2}}$$

$$\frac{d\theta}{dt} = \frac{4\sqrt{3}}{(t+2)^2 + 3(t-2)^2}$$

(a) 4 hours after the start, t = 4

$$\frac{d\theta}{dt} = \frac{4\sqrt{3}}{(4+2)^2 + 3(4-2)^2}$$

$$\frac{d\theta}{dt} = \frac{4\sqrt{3}}{36 + 12}$$

$$\frac{d\theta}{dt} = \frac{4\sqrt{3}}{48}$$

$$\frac{d\theta}{dt} = \frac{1}{12}\sqrt{3}\ \text{rad/sec}$$

answer

(b) Just after the turn, t = 2

$$\frac{d\theta}{dt} = \frac{4\sqrt{3}}{(2+2)^2 + 3(2-2)^2}$$

$$\frac{d\theta}{dt} = \frac{4\sqrt{3}}{16+0}$$

$$\frac{d\theta}{dt} = \frac{4\sqrt{3}}{16}$$

$$\frac{d\theta}{dt} = \frac{1}{4}\sqrt{3}\ \text{rad/sec}$$

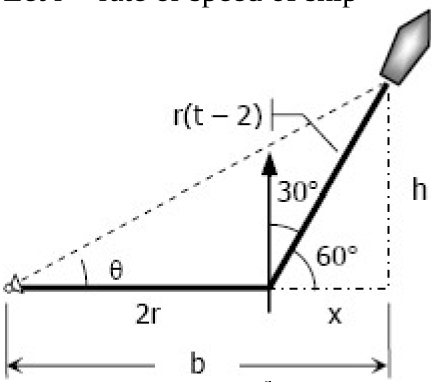
answer

Problem 51

Show that the answers to Problem 49 are independent of the speed of the ship.

Solution 51

Let r = rate or speed of ship



$$h = r(t-2)\sin 60°$$

$$h = \frac{1}{2}\sqrt{3}r(t-2)$$

$$x = r(t-2)\cos 60°$$

$$x = \frac{1}{2}(t-2)$$

$$b = 2r + x = 2r + \frac{1}{2}r(t-2)$$

$$b = 2r + \frac{1}{2}rt - r = \frac{1}{2}rt + r$$

$$b = \frac{1}{2}r(t+2)$$

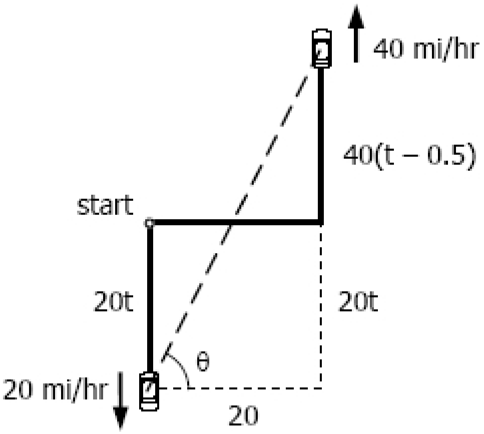
$$\theta = \arctan\frac{h}{b} = \arctan\frac{\frac{1}{2}\sqrt{3}r(t-2)}{\frac{1}{2}r(t+2)}$$

$$\theta = \arctan\frac{\sqrt{3}(t-2)}{t+2}$$

The above value of angle θ did not include the speed of the ship r. The derivative of θ in terms of time t will of course be independent from r. Problem 49 is a derivative of θ in terms of time t.

Problem 52
A car drives south at 20 mi/hr. Another car, starting from the same point at the same time and traveling 40 mi/hr, goes east for 30 minutes then turns north. Find the rate of rotation of the line joining the cars (a) 1 hour after the start; (b) at the time the second car makes its turn.

Solution 52



$$\tan\theta = \frac{20t + 40(t - 0.5)}{20}$$

$$\tan\theta = t + 2(t - 0.5)$$

$$\tan\theta = 3t - 1$$

$$\theta = \arctan(3t - 1)$$

$$\frac{d\theta}{dt} = \frac{3}{1 + (3t - 1)^2}$$

(a) 1 hour after the start, t = 1

$$\frac{d\theta}{dt} = \frac{3}{1 + [3(1) - 1]^2}$$

$$\frac{d\theta}{dt} = \frac{3}{1 + 4}$$

$$\frac{d\theta}{dt} = 0.6 \text{ rad/hr}$$

answer

(b) At the time the second car makes its turn, t = 0.5

$$\frac{d\theta}{dt} = \frac{3}{1 + [3(0.5) - 1]^2}$$

$$\frac{d\theta}{dt} = \frac{3}{1 + 0.25}$$

$$\frac{d\theta}{dt} = \frac{3}{1 + 0.25}$$

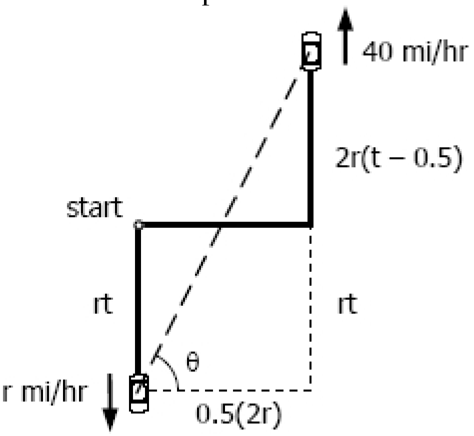
$$\frac{d\theta}{dt} = 2.4 \text{ rad/hr}$$

answer

Problem 53
Prove that the results in Problem 52 are independent of the speed of the cars, if the second car travels twice as fast as the first car.

Solution 53

Let r = rate or speed of first car



2r = rate or speed of the second car

$$\tan\theta = \frac{rt + 2r(t - 0.5)}{0.5(2r}$$

$$\tan\theta = t + 2(t - 0.5)$$

$$\tan\theta = 3t - 1$$

$$\theta = \arctan(3t - 1)$$

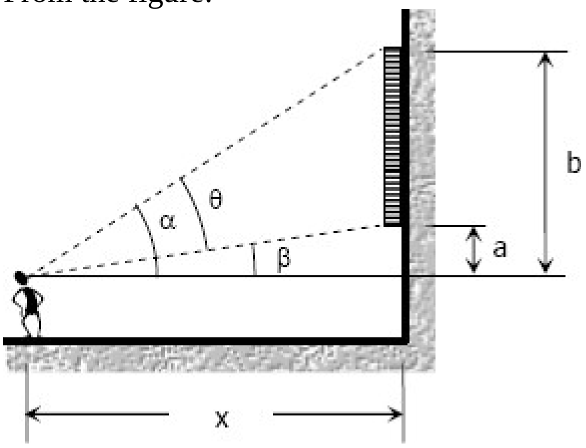
The angle θ above is independent from rate of cars, r. *answer*

Problem 55

The lower edge of the picture is a ft, the upper edge is b ft, above the eye of an observer. At what horizontal distance should he stand, if the vertical angle subtended by the picture is to be greatest?

Solution 55

From the figure:



$$\theta = \alpha - \beta$$

$$\tan\theta = \tan(\alpha - \beta)$$

$$\tan\theta = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\,\tan\beta}$$

$$\tan\theta = \frac{\frac{b}{x} - \frac{a}{x}}{1 + \frac{b}{x}\left(\frac{a}{x}\right)}$$

$$\tan\theta = \frac{\frac{b-a}{x}}{\frac{x^2+ab}{x^2}}$$

$$\tan\theta = \frac{b-a}{\frac{x^2+ab}{x}}$$

$$\tan\theta = \frac{(b-a)x}{x^2+ab}$$

$$\theta = \arctan\frac{(b-a)x}{x^2+ab}$$

$$\frac{d\theta}{dx} = \frac{\dfrac{(x^2+ab)(b-a)-(b-a)x(2x)}{(x^2+ab)^2}}{1+\left[\dfrac{(b-a)x}{x^2+ab}\right]^2} = 0$$

$$\frac{(x^2+ab)(b-a)-(b-a)x(2x)}{(x^2+ab)^2} = 0$$

$$(x^2+ab)(b-a)-(b-a)x(2x) = 0$$

$$(b-a)\,[\,(x^2+ab)-2x^2\,] = 0$$

$$(x^2+ab)-2x^2 = 0$$

$$ab - x^2 = 0$$

$$ab = x^2$$

$$x = \sqrt{ab}$$

answer