Tutorial Solution
Chapter 1: Linear Equations and Matrices

1. Given the linear system

$$\begin{array}{rcrcl} 2x & - & y & = & 5 \\ 4x & - & 2y & = & t \end{array}.$$

(a) Determine a particular value of t so that the system is consistent.

(b) Determine a particular value of t so that the system is inconsistent.

(c) How many different values of t can be selected in part (b)?

Solution

Rewrite both equations as

$$\begin{array}{rcl} y & = & 2x - 5, \\ y & = & 2x - \dfrac{t}{2}. \end{array}$$

Note that both equations represent straight lines and the slope for both equations is the same, i.e. both are parallel lines. The only difference is the y-intercept.

(a) A system is consistent if it has unique or infinitely solution. The only way for the linear system to be consistent is to let both parallel lines touch each other. Therefore $t = 10$.

(b) A system is inconsistent when both lines do not intersect each other. So one value for t is 3.

(c) Any choice for t, other than $t = 10$, makes both lines do not cross each other and thus, makes the system inconsistent. Therefore, there are infinitely many ways to choose a value for t in part (b).

2. Given the linear system

$$\begin{array}{rcrcl} x & + & 2y & = & 10 \\ 3x & + & (6+t)y & = & 30 \end{array}.$$

(a) Determine a particular value of t so that the system has infinitely many solutions.

(b) Determine a particular value of t so that the system has a unique solution.

(c) How many different values of t can be selected in part (b)?

Solution
Rewrite both equations as

$$\begin{aligned} y &= -\frac{x}{2} + 5, \\ y &= -\frac{3x}{6+t} + \frac{30}{6+t}. \end{aligned}$$

(a) In order to make the system to have infinitely many solutions, both lines must be parallel and touching each other, i.e. both equations must have the same slope and y-intercept. So we have

$$\begin{aligned} -\frac{3}{6+t} &= -\frac{1}{2}, \\ t &= 0. \end{aligned}$$

Note that when $t = 0$, $\frac{30}{6+t} = 5$. Therefore, $t = 0$.

(b) One value is $t = 1$ such that the second equation becomes $y = -\frac{3}{7}x + \frac{30}{7}$ and thus both lines will intersect at one point, which is our unique solution.

(c) Any value for t as long as $t \neq 0$.

3. If $\begin{bmatrix} a+2b & 2a-b \\ 2c+d & c-2d \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 4 & -3 \end{bmatrix}$, find a, b, c and d.

Solution
From the equation, we obtain two set of linear system:

$$\left.\begin{aligned} a + 2b = 4 \\ 2a - b = -2 \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} 2c + d = 4 \\ c - 2d = -3 \end{aligned}\right\} \quad (2)$$

Solving both set of equations simultaneously, we obtain $a = 0, b = 2, c = 1, d = 2$.

4. Consider the following matrices for the following question.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 3 & 4 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -1 \\ -2 & 3 \\ 4 & 5 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, \qquad D = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & -3 & 4 \\ 1 & -2 & 3 \\ 4 & 1 & 0 \end{bmatrix}, \qquad F = \begin{bmatrix} -1 & 2 \\ 0 & 4 \\ 3 & 5 \end{bmatrix}.$$

If possible, compute the following:

(a) AB (c) F^TE (e) $AB + D^2$

(b) BA (d) $CB + D$ (f) $2F - 3(AE)$

Solution

(a) $\begin{bmatrix} -4 & -9 \\ 21 & 34 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -1 & -11 \\ 5 & 10 & 19 \\ 23 & 24 & 17 \end{bmatrix}$

(c) $\begin{bmatrix} 10 & 6 & -4 \\ 28 & -9 & 20 \end{bmatrix}$

(d) Impossible

(e) $\begin{bmatrix} 3 & -7 \\ 27 & 53 \end{bmatrix}$

(f) Impossible

5. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Compute the following entries of AB:

(a) the $(1,2)$ entry (b) the $(2,3)$ entry (c) the $(3,1)$ entry (d) the $(3,3)$ entry

Solution

(a) $c_{12}(\text{row}_1(A))^T \cdot \text{col}_2(B) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 4.$

(b) $c_{23}(\text{row}_2(A))^T \cdot \text{col}_3(B) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 13.$

(c) $c_{31}(\text{row}_3(A))^T \cdot \text{col}_1(B) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3.$

(d) $c_{33}(\text{row}_3(A))^T \cdot \text{col}_3(B) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 12.$

6. Write the linear system whose augmented matrix is

$$\left[\begin{array}{rrrr|r} -2 & -1 & 0 & 4 & 5 \\ -3 & 2 & 7 & 8 & 3 \\ 1 & 0 & 0 & 2 & 4 \\ 3 & 0 & 1 & 3 & 6 \end{array}\right].$$

Solution

$$\begin{array}{rcrcrcrcr} -2x_1 & - & x_2 & & & + & 4x_4 & = & 5 \\ -3x_1 & + & 2x_2 & + & 7x_3 & + & 8x_4 & = & 3 \\ x_1 & & & & & + & 2x_4 & = & 4 \\ 3x_1 & & & + & x_3 & + & 3x_4 & = & 6 \end{array}.$$

7. Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$. Compute each of the following:

(a) A^3 (b) B^2 (c) $(AB)^3$

Solution

(a) $A^3 = A \cdot A \cdot A = \begin{bmatrix} 3 & 0 & 1 \\ 5 & 8 & -1 \\ 2 & 4 & 2 \end{bmatrix}$

(b) $B^2 = B \cdot B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

(c) $(AB)^3 = (AB) \cdot (AB) \cdot (AB) = \begin{bmatrix} -36 & 12 & 8 \\ 4 & 20 & 104 \\ -16 & 24 & 96 \end{bmatrix}$

8. If $A^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 5 \\ 3 & -2 \end{bmatrix}$, find $(AB)^{-1}$.

Solution

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 2 & 5 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 0 & -5 \end{bmatrix}$$