DSA/ISE 5013: Fundamentals of Engineering Statistical Analysis

ISE 3293: Applied Engineering Statistics

Lecture 8: Binomial and Geometric Distributions

September 16, 2019

Discussion

- **HW1 Solutions**

Discussion: Discrete Random Variables

- What is a random variable? Why is it important?
- What is cumulative distribution function or probability distribution?
- What is the expected value?
- What is the variance?

Learning Objectives

- Revisit counting methods: permutations and combinations
- Understand the assumptions for binomial and geometric distributions
- Select an appropriate discrete probability distribution to calculate probabilities in specific applications

Review: Counting and Permutation

A combination lock has 40 numbers on it.

(a) How many different three-number combinations can be made?

(b) How many different combinations are there if the numbers must be all different?

Review: Counting and Permutation

A **permutation** is an arrangement of all or part of a set of objects.

Definition

For any non-negative integer n, $n!$, called "n factorial," is defined as

$$n! = n(n-1)\cdots 3 \cdot 2 \cdot 1,$$

with special case $0! = 1$.

Permutations (with a twist)

Theorem

The number of permutations of n distinct objects taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!}$$

Permutation: Horse racing

In horse racing, a trifecta is a type of bet. To win a trifecta bet, you need to specify the horses that finish in the top three spots in the exact order in which they finish. If eight horses enter the race, how many different ways can they finish in the top three spots?

Selecting objects without ordering

How many ways can two slices of pizza be chosen from a plate containing one slice each of pepperoni, sausage, mushroom, and cheese pizza.

Combinations!

Theorem

The number of combinations of n distinct objects taken r at a time is

$$C_n^k \equiv \left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{r!(n-r)!}$$

Comparing combinations and permutations

- **Combination implies SELECTION**
- **Permutation implies ARRANGEMENT**

Now What?

Notes

- Some discrete random variables are described with functional forms
- Some of these functional forms require the use of counting methods
- We will discuss the Binomial, Geometric, Hypergeometric, and Poisson distributions

Note

- For all distribution problems
  - Define the random variable, which helps us translate a problem into a solution)
  - Find probability distribution function
  - Calculate the mean and variance

Bernoulli Random Variable

Definition

Any random variable whose only possible values are 0 and 1 is called a Bernoulli random variable.

Since a Bernoulli trial has only two possible outcomes, it can be framed as some "yes or no" question.

Example

- Did the coin land heads?
- Was the newborn child a girl?

Bernoulli trials

- Bernoulli(p) $= \begin{cases} 1 & \text{(success) with probability p,} \\ 0 & \text{(failure) with probability 1 - p.} \end{cases}$

Binomial Distribution

- binomial(n,p) = # of "successes" in
  n independent Bernoulli(p) trials.

Counting...

- count number of "yes"-answers in a poll
- count number of defective parts on an assembly line
- count number of points after 5 free throws (basketball)
- count number of ...

Binomial Distribution: Find the pmf

- binomial(n,p) = # of "successes" in
  n independent Bernoulli(p) trials.

Binomial Distribution

- X =binomial(n,p)

$$P(X = x) \equiv b(x, n, p) = C_x^n p^x (1-p)^{n-x}, \ x = 0, 1, \ldots, n$$

x	b(x,n,p)
0	$(1-p)^n$
1	$n \cdot p(1-p)^{n-1}$
2	$C_n^2 p^2 (1-p)^{n-2}$
3	$C_n^3 p^3 (1-p)^{n-3}$
$\ldots$	$\ldots$
n	p^n

Example: Binomial distribution

Bob plays basketball. Bob has a 57.2% chance of making a regular shot. However, he makes only 51.3% of his foul shots. Regular shots are worth two points. Foul shots are worth one point each.

- Calculate the expected value (EV) of the number of points Bob scores on one regular shot (not foul shots) at the basket (i.e., he makes or misses the shot).

Example: Binomial distribution

Bob plays basketball. Bob has a 57.2% chance of making a regular shot. However, he makes only 51.3% of his foul shots. Regular shots are worth two points. Foul shots are worth one point each.

- Assume that all foul shots are independent events (examinations of foul shooting records suggest this is approximately true). Calculate the expected value (EV) of the number of points Bob gets when he shoots two foul shots.

Where does the name binomial come from?

The binomial distribution derives its name from the fact that the $n+1$terms in the binomial expansion of $(q+p)^n$ corresponds to various values of $b(x;n,p)$ for $x=0,1,2,...,n$. That is,

$$(q+p)^n = \left(\begin{array}{c} n \\ 0 \end{array}\right) q^n + \left(\begin{array}{c} n \\ 1 \end{array}\right) pq^{n-1} + \left(\begin{array}{c} n \\ 2 \end{array}\right) p^2q^{n-2} + ... + \left(\begin{array}{c} n \\ n \end{array}\right) p^n$$

Since $p+q=1$, we see that $\sum_{x=0}^{n} b(x;n,p) = 1$. A condition that must hold for any probability distribution.

Binomial Distribution

Situations leading to Binomial Distributions:

- Fixed number of independent trials
- On each trial, two mutually exclusive events can happen, and we are interested in the number of occurrences.
- Probabilities for each of the events remain constant

The mean and variance of the binomial distribution $b(x; n, p)$ are:

- $E(X) = \mu = np$
- $V(X) = \sigma^2 = npq$
- Proof ?

Decisive Family

A family decides to have babies until the first girl is born!

Assumptions:

- probability of having a girl with each pregnancy is 0.5 independent of other pregnancies
- all babies survive

What is the probability that a family has k boys?

Decisive Family

Geometric distribution vs Bernoulli

geometric(p) $=$ # of trials before the first success in a sequence of Bernoulli(p) trials.

Geometric distribution

Definition

A random variable X has a geometric distribution with parameter p if

- it takes the values 1, 2, 3, ...
- its pmf is given by $P(X = k) = (1 - p)^{k-1} p$.

We denote this by $X \sim$ geometric(p) or geo(p).

Geometric Distribution: Properties

If a random variable X has a geometric distribution with parameter p, then its expected value and variance are

- $E[X] = \frac{1}{p}$
- $V(X) = \frac{1-p}{p^2}$
- Proof ?