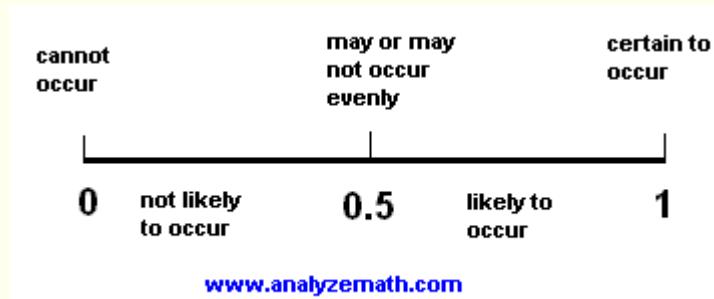


Introduction to Probability

Probabilities are associated with experiments where the outcome is not known in advance or cannot be predicted. For example, if you toss a coin, will you obtain a head or tail? If you roll a die will obtain 1, 2, 3, 4, 5 or 6? Probability measures and quantifies "how likely" an event, related to these types of experiment, will happen. The value of a probability is a number between 0 and 1 inclusive. An event that cannot occur has a probability (of happening) equal to 0 and the probability of an event that is certain to occur has a probability equal to 1.(see probability scale below).



In order to quantify probabilities, we need to define the **sample space** of an experiment and the **events** that may be associated with that experiment.

Sample Space and Events

The **sample space** is the set of all possible outcomes in an experiment.

Example 1: If a die is rolled, the sample space S is given by

$$S = \{1,2,3,4,5,6\}$$

Example 2: If two coins are tossed, the sample space S is given by

$$S = \{HH,HT,TH,TT\}, \text{ where } H = \text{head and } T = \text{tail.}$$

Example 3: If two dice are rolled, the sample space S is given by

$$S = \{ (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \}$$

We define an **event** as some specific outcome of an experiment. An event is a subset of the sample space.

Example 4: A die is rolled (see example 1 above for the sample space). Let us define event E as the set of possible outcomes where the number on the face of the die is even. Event E is given by

$$E = \{2,4,6\}$$

Example 5: Two coins are tossed (see example 2 above for the sample space). Let us define event E as the set of possible outcomes where the number of head obtained is equal to two. Event E is given by

$$E = \{(HT),(TH)\}$$

Example 6: Two dice are rolled (see example 3 above for the sample space). Let us define event E as the set of possible outcomes where the sum of the numbers on the faces of the two dice is equal to four. Event E is given by

$$E = \{(1,3),(2,2),(3,1)\}$$

How to Calculate Probabilities?

1 - Classical Probability Formula: It is based on the fact that all outcomes are equally likely.

$$P(E) = \frac{\text{Total number of outcomes in E}}{\text{Total number of outcomes in the sample space}}$$

Example 7: A die is rolled, find the probability of getting a 3.

The event of interest is "getting a 3". so $E = \{3\}$.

The sample space S is given by $S = \{1,2,3,4,5,6\}$.

The number of possible outcomes in E is 1 and the number of possible outcomes in S is 6. Hence the probability of getting a 3 is $P("3") = 1 / 6$.

Example 8: A die is rolled, find the probability of getting an even number.

The event of interest is "getting an even number". so $E = \{2,4,6\}$, the even numbers on a die.

The sample space S is given by $S = \{1,2,3,4,5,6\}$.

The number of possible outcomes in E is 3 and the number of possible outcomes in S is 6. Hence the probability of getting a 3 is $P("3") = 3 / 6 = 1 / 2$.

2 - Empirical Probability Formula: It uses real data on present situations to determine how likely outcomes will occur in the future. Let us clarify this using an example

30 people were asked about the colors they like and here are the results:

Color	frequency
red	10
blue	15
green	5

If a person is selected at random from the above group of 30, what is the probability that this person likes the red color?

Let event E be "likes the red color". Hence

$$P(E) = \frac{\text{Frequency for red color}}{\text{Total frequencies in the above table}}$$
$$= 10 / 30 = 1 / 3$$

Example 8: The table below shows students distribution per grade in a school.

Grade	frequency
1	50
2	30
3	40
4	42
5	38
6	50

If a student is selected at random from this school, what is the probability that this student is in grade 3?

Let event E be "student from grade 3". Hence

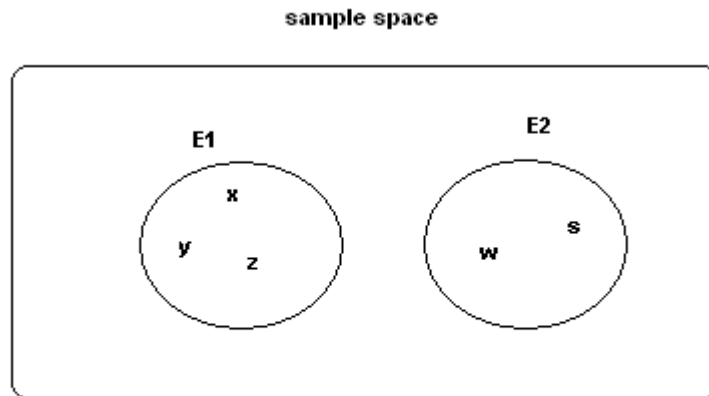
$$P(E) = \frac{\text{Frequency for grade 3}}{\text{Total frequencies}}$$
$$= 40 / 250 = 0.16$$

Mutually Exclusive Events - Examples With Solutions

A quick review of [sample space](#) of an experiment and the [events](#) related to a sample space may be necessary.

Two events are **mutually exclusive** if they cannot occur at the same time.

Using Venn diagram, two events that **are mutually exclusive** may be represented as follows:



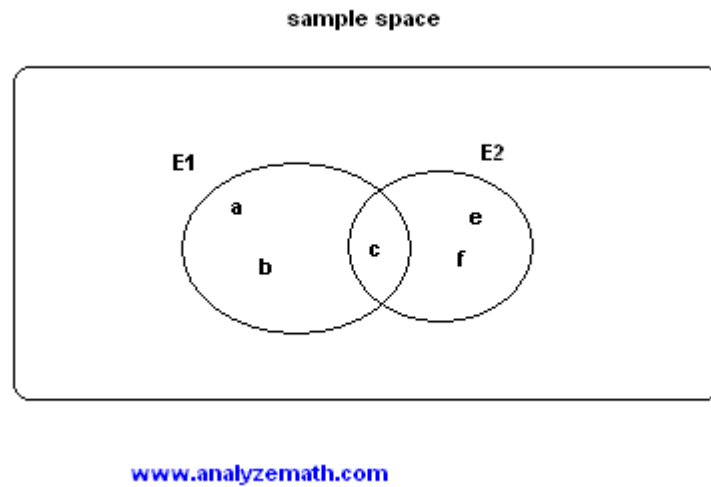
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The two events are such that

$$E1 \cap E2 = \Phi$$

The two sets E1 and E2 have no elements in common and their intersection is an empty set since they cannot occur at the same time.

Using Venn diagram, two events that **are not mutually exclusive** may be represented as follows:



$E1 \cap E2 = \{c\}$, the intersection of the two events E1 and E2 is not an empty set

Example 1:

A die is rolled. Let us define event E1 as the set of possible outcomes where the number on the face of the die is even and event E2 as the set of possible outcomes where the number on the face of the die is odd. Are event1 E1 and E2 mutually exclusive?

Solution to Example 1:

- We first list the elements of E1 and E2.

$$E1 = \{2,4,6\}$$

$$E2 = \{1,3,5\}$$

- E1 and E2 have no elements in common and therefore are mutually exclusive.
- Another way to answer the above question is to note is that if you roll a die, it shows a number that is either even or odd but no number will be even and odd at the same time. Hence E1 and E2 cannot occur at the same time and are therefore mutually exclusive.

Example 2:

A die is rolled. Event E1 is the set of possible outcomes where the number on the face of the die is even and event E2 as the set of possible outcomes where the number on the face of the die is greater than 3. Are event E1 and E2 mutually exclusive?

Solution to Example 2:

- The subsets E1 and E2 are given by.

$$E1 = \{2,4,6\}$$

$$E2 = \{4,5,6\}$$

- Subsets E1 and E2 have 2 elements in common. If the die shows 4 or 6, both events E1 and E2 will have occurred at the same time and therefore E1 and E2 are not mutually exclusive.

Example 3:

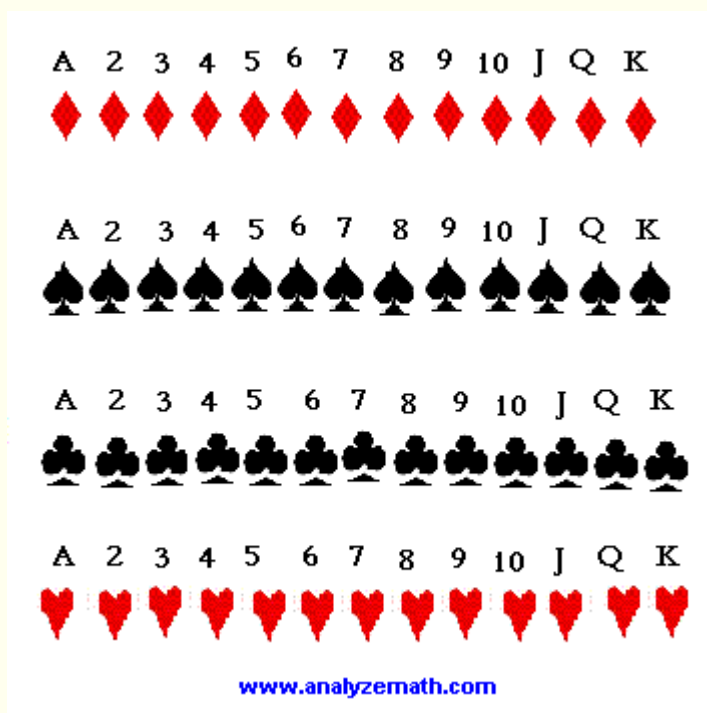
A card is drawn from a deck of cards. Events E1, E2, E3, E4 and E5 are defined as follows:

- E1: Getting an 8
- E2: Getting a king
- E3: Getting a face card
- E4: Getting an ace
- E5: Getting a heart

- a) Are events E1 and E2 mutually exclusive?
- b) Are events E2 and E3 mutually exclusive?
- c) Are events E3 and E4 mutually exclusive?
- d) Are events E4 and E5 mutually exclusive?
- e) Are events E5 and E1 mutually exclusive?

Solution to Example 3:

- The sample space of the experiment "card is drawn from a deck of cards" is shown below.



- a) E1 and E2 are **mutually exclusive** because there are no cards with an 8 and a king together.
- b) E2 and E3 are **not mutually exclusive** because a king is a face card.
- c) E3 and E4 are **mutually exclusive** because an ace is not a face card.
- d) E4 and E5 are **not mutually exclusive** because there is one card that has an ace and a heart.
- d) E5 and E1 are **not mutually exclusive** because there is one card that is an 8 of heart.

Example 4: Two dice are rolled. We define events E1, E2, E3 and E4 as follows

E1: Getting a sum equal to 10

E2: Getting a double

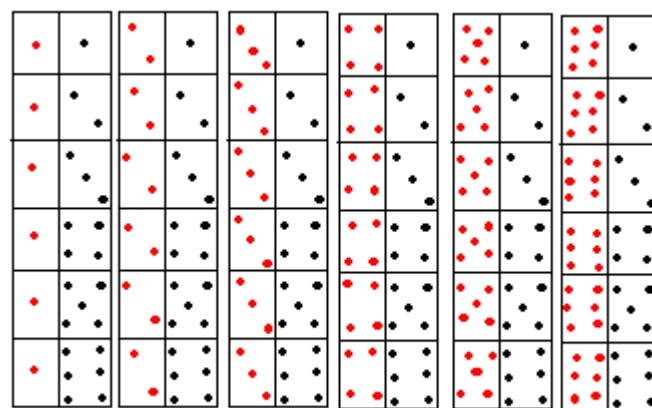
E3: Getting a sum less than 4

E4: Getting a sum less to 7

- Are events E1 and E2 mutually exclusive?
- Are events E2 and E3 mutually exclusive?
- Are events E3 and E4 mutually exclusive?
- Are events E4 and E1 mutually exclusive?

Solution to Example 4:

- The sample space of the experiment "2 dice" is shown below.



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- a) E1 and E2 are **not mutually exclusive** because outcome (5,5) is a double and also gives a sum of 10. The two events may occur at the same time.

- b) E2 and E3 are **not mutually exclusive** because outcome (1,1) is a double and gives a sum of 2 and is less than 4. The two events E2 and E3 may occur at the same time.
- c) E3 and E4 are **not mutually exclusive** a sum can be less than 7 and less than 4 at the same time. Example outcome (1,2).
- d) E4 and E1 are **mutually exclusive** because a sum less than 7 cannot be equal to 10 at the same time. The two events cannot occur at the same time.

Probability Questions with Solutions

In what follows, S is the [sample space](#) of the experiment in question and E is the [event](#) of interest. $n(S)$ is the number of elements in the sample space S and $n(E)$ is the number of elements in the event E.

Question 1: A die is rolled, find the probability that an even number is obtained.

Solution to Question 1:

- Let us first write the [sample space](#) S of the experiment.

$$S = \{1,2,3,4,5,6\}$$

- Let E be the [event](#) "an even number is obtained" and write it down.

$$E = \{2,4,6\}$$

- We now use the formula of the [classical](#) probability.

$$P(E) = n(E) / n(S) = 3 / 6 = 1 / 2$$

Question 2: Two coins are tossed, find the probability that two heads are obtained.

Note: Each coin has two possible outcomes H (heads) and T (Tails).

Solution to Question 2:

- The sample space S is given by.

$$S = \{(H,T),(H,H),(T,H),(T,T)\}$$

- Let E be the event "two heads are obtained".

$$E = \{(H,H)\}$$

- We use the formula of the classical probability.

$$P(E) = n(E) / n(S) = 1 / 4$$

Question 3: Which of these numbers cannot be a probability?

- a) -0.00001
- b) 0.5
- c) 1.001
- d) 0
- e) 1
- f) 20%

Solution to Question 3:

- A probability is always greater than or equal to 0 and less than or equal to 1, hence only **a)** and **c)** above cannot represent probabilities: -0.00010 is less than 0 and 1.001 is greater than 1.

Question 4: Two dice are rolled, find the probability that the sum is

- a) equal to 1
- b) equal to 4
- c) less than 13

Solution to Question 4:

- **a)** The sample space S of two dice is shown below.

$$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6) \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6) \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6) \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

- Let E be the event "sum equal to 1". There are no outcomes which correspond to a sum equal to 1, hence

$$P(E) = n(E) / n(S) = 0 / 36 = 0$$

- b) Three possible outcomes give a sum equal to 4: $E = \{(1,3),(2,2),(3,1)\}$, hence.

$$P(E) = n(E) / n(S) = 3 / 36 = 1 / 12$$

- c) All possible outcomes, $E = S$, give a sum less than 13, hence.

$$P(E) = n(E) / n(S) = 36 / 36 = 1$$

Question 5: A die is rolled and a coin is tossed, find the probability that the die shows an odd number and the coin shows a head.

Solution to Question 5:

- The sample space S of the experiment described in question 5 is as follows

$$S = \{ (1,H),(2,H),(3,H),(4,H),(5,H),(6,H) \\ (1,T),(2,T),(3,T),(4,T),(5,T),(6,T) \}$$

- Let E be the event "the die shows an odd number and the coin shows a head". Event E may be described as follows

$$E = \{(1,H),(3,H),(5,H)\}$$

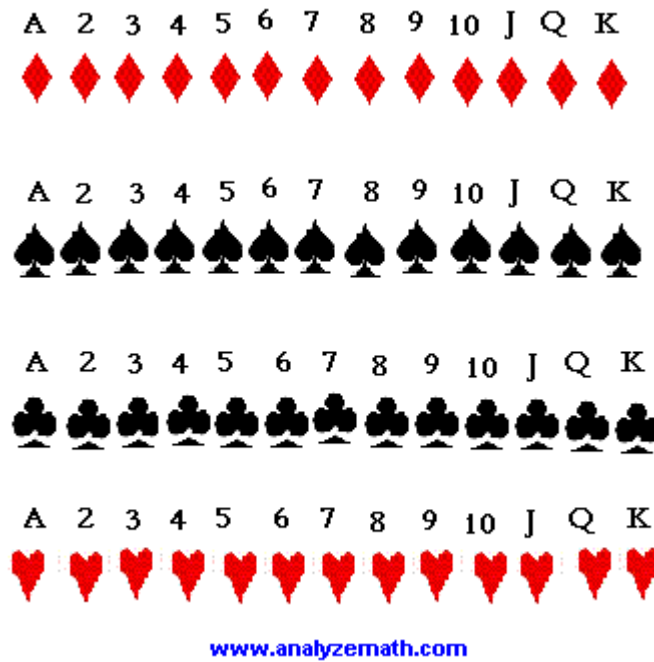
- The probability $P(E)$ is given by

$$P(E) = n(E) / n(S) = 3 / 12 = 1 / 4$$

Question 6: A card is drawn at random from a deck of cards. Find the probability of getting the 3 of diamond.

Solution to Question 6:

- The sample space S of the experiment in question 6 is shown below



- Let E be the event "getting the 3 of diamond". An examination of the sample space shows that there is one "3 of diamond" so that $n(E) = 1$ and $n(S) = 52$. Hence the probability of event E occurring is given by

$$P(E) = 1 / 52$$

Question 7: A card is drawn at random from a deck of cards. Find the probability of getting a queen.

Solution to Question 7:

- The sample space S of the experiment in question 7 is shown above (see question 6)
- Let E be the event "getting a Queen". An examination of the sample space shows that there are 4 "Queens" so that $n(E) = 4$ and $n(S) = 52$. Hence the probability of event E occurring is given by

$$P(E) = 4 / 52 = 1 / 13$$

Question 8: A jar contains 3 red marbles, 7 green marbles and 10 white marbles. If a marble is drawn from the jar at random, what is the probability that this marble is white?

Solution to Question 8:

- We first construct a table of frequencies that gives the marbles color distributions as follows

color frequency

red 3
green 7
white 10

-
- We now use the empirical formula of the probability

$$P(E) = \frac{\text{Frequency for white color}}{\text{Total frequencies in the above table}}$$

-

$$= 10 / 20 = 1 / 2$$

Question 9: The blood groups of 200 people is distributed as follows: 50 have type **A** blood, 65 have **B** blood type, 70 have **O** blood type and 15 have type **AB** blood. If a person from this group is selected at random, what is the probability that this person has O blood type?

Solution to Question 9:

- We construct a table of frequencies for the the blood groups as follows

group	frequency
a	50
B	65
O	70
AB	15

-
- We use the empirical formula of the probability

$$P(E) = \frac{\text{Frequency for O blood}}{\text{Total frequencies}}$$

-

$$= 70 / 200 = 0.35$$

Exercises:

- A die is rolled, find the probability that the number obtained is greater than 4.
- Two coins are tossed, find the probability that one head only is obtained.
- Two dice are rolled, find the probability that the sum is equal to 5.

d) A card is drawn at random from a deck of cards. Find the probability of getting the King of heart.

Answers to above exercises:

a) $2 / 6 = 1 / 3$

b) $2 / 4 = 1 / 2$

c) $4 / 36 = 1 / 9$

d) $1 / 52$

Tutorial on Discrete Probability Distributions

Let X be a random variable that takes the numerical values X_1, X_2, \dots, X_n with probabilities $p(X_1), p(X_2), \dots, p(X_n)$ respectively. A discrete probability distribution consists of the values of the random variable X and their corresponding probabilities $P(X)$.

The probabilities $P(X)$ are such that

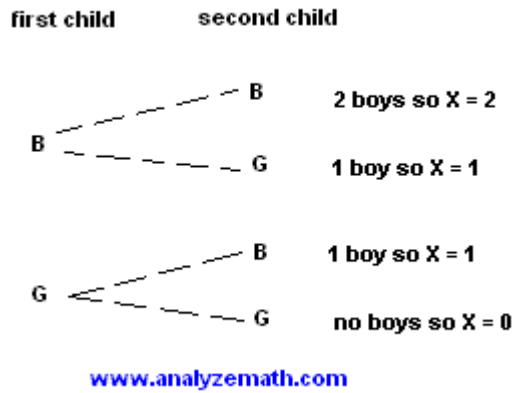
$$\sum P(X) = 1$$

Example 1: Let the random variable X represents the number of boys in a family.

- a) Construct the probability distribution for a family of two children.
- b) Find the mean and standard deviation of X .

Solution to Example 1:

- a) We first construct a tree diagram to represent all possible distributions of boys and girls in the family.



- Assuming that all the above possibilities are equally likely, the probabilities are:

$$P(X=2) = P(BB) = 1 / 4$$

$$P(X=1) = P(BG) + P(GB) = 1 / 4 + 1 / 4 = 1 / 2$$

$$P(X=0) = P(GG) = 1 / 4$$

- The discrete probability distribution of X is given by

X	P(X)
0	1 / 4
1	1 / 2
2	1 / 4

- Note that

$$\sum P(X) = 1$$

- b) The mean μ of the random variable X is defined by

$$\mu = \sum X P(X)$$

$$= 0 * (1/4) + 1 * (1/2) + 2 * (1/4) = 1$$

- The standard deviation σ of the random variable X is defined by

$$\sigma = \text{Square Root} [\sum (X - \mu)^2 P(X)]$$

$$= \text{Square Root} [(0 - 1)^2 * (1/4) + (1 - 1)^2 * (1/2) + (2 - 1)^2 * (1/4)]$$

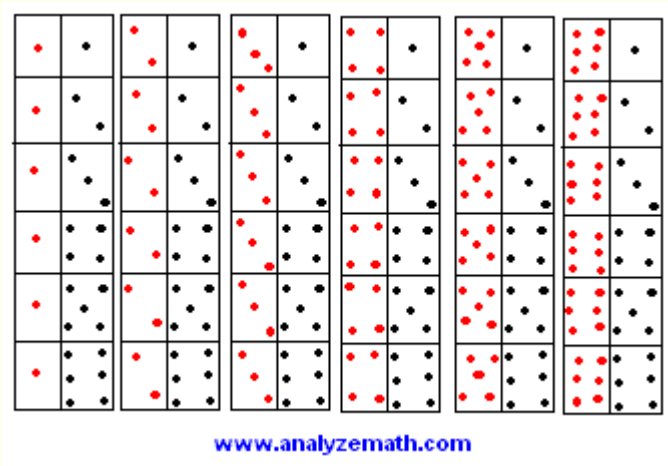
$$= 1 / \text{square root} (2)$$

Example 2: Two balanced dice are rolled. Let X be the sum of the two dice.

- Obtain the probability distribution of X.
- Find the mean and standard deviation of X.

Solution to Example 2:

- a) When the two balanced dice are rolled, there are 36 equally likely possible outcomes as shown below .



- The possible values of X are: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12.
- The possible outcomes are equally likely hence the probabilities P(X) are given by

$$P(2) = P(1,1) = 1 / 36$$

$$P(3) = P(1,2) + P(2,1) = 2 / 36 = 1 / 18$$

$$P(4) = P(1,3) + P(2,2) + P(3,1) = 3 / 36 = 1 / 12$$

$$P(5) = P(1,4) + P(2,3) + P(3,2) + P(4,1) = 4 / 36 = 1 / 9$$

$$P(6) = P(1,5) + P(2,4) + P(3,3) + P(4,2) + P(5,1) = 5 / 36$$

$$P(7) = P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1) = 6 / 36 = 1 / 6$$

$$P(8) = P(2,6) + P(3,5) + P(4,4) + P(5,3) + P(6,2) = 5 / 36$$

$$P(9) = P(3,6) + P(4,5) + P(5,4) + P(6,3) = 4 / 36 = 1 / 9$$

$$P(10) = P(4,6) + P(5,5) + P(6,4) = 3 / 36 = 1 / 12$$

$$P(11) = P(5,6) + P(6,5) = 2 / 36 = 1 / 18$$

$$P(12) = P(6,6) = 1 / 36$$

- The discrete probability distribution of X is given by

X	P(X)
2	1 / 36
3	1 / 18
4	1 / 12
5	1 / 9
6	5 / 36
7	1 / 6
8	5 / 36
9	1 / 9
10	1 / 12
11	1 / 18
12	1 / 36

-
- As an exercise, check that

$$\sum P(X) = 1$$

- b) The mean of X is given by

$$\begin{aligned} \mu &= \sum X P(X) \\ &= 2*(1/36)+3*(1/18)+4*(1/12)+5*(1/9)+6*(5/36) \\ &\quad +7*(1/6)+8*(5/36)+9*(1/9)+10*(1/12) \\ &\quad +11*(1/18)+12*(1/36) \\ &= 7 \end{aligned}$$

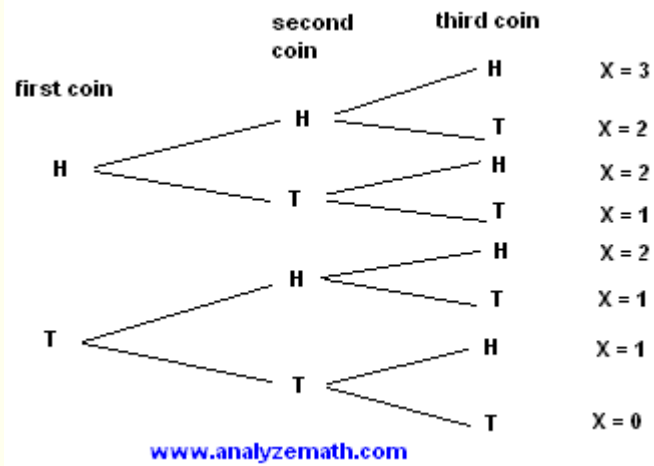
- The standard deviation of is given by

$$\begin{aligned} \mu &= \sigma \text{ Square Root } [\sum (X - \mu)^2 P(X)] \\ &= \text{Square Root } [(2-7)^2*(1/36)+(3-7)^2*(1/18) \\ &\quad +(4-7)^2*(1/12)+(5-7)^2*(1/9)+(6-7)^2*(5/36) \\ &\quad +(7-7)^2*(1/6)+(8-7)^2*(5/36)+(9-7)^2*(1/9) \\ &\quad +(10-7)^2*(1/12)+(11-7)^2*(1/18)+(12-7)^2*(1/36)] \\ &= 2.41 \end{aligned}$$

Example 3: Three coins are tossed. Let X be the number of heads obtained. Construct a probability distribution for X and find its mean and standard deviation.

Solution to Example 3:

- The tree diagram representing all possible outcomes when three coins are tossed is shown below.



- Assuming that all three coins are identical and all possible outcomes are equally likely, the probabilities are:

$$P(X=0) = P(TTT) = 1 / 8$$

$$P(X=1) = P(HTT) + P(THT) + P(TTH)$$

$$= 1 / 8 + 1 / 8 + 1 / 8$$

$$= 3 / 8$$

$$P(X=2) = P(HHT) + P(HTH) + P(THH)$$

$$= 1 / 8 + 1 / 8 + 1 / 8$$

$$= 3 / 8$$

$$P(X=3) = P(HHH) = 1 / 8$$

- The discrete probability distribution of X is given by

X	P(X)
0	1 / 8
1	3 / 8
2	3 / 8
3	1 / 8

- Note that

$$\sum P(X) = 1$$

- We now compute the mean μ of the random variable X as follows

$$\mu = \sum X P(X)$$

$$= 0 * (1/8) + 1 * (3/8) + 2 * (3/8) + 3 * (1/8) = 1.5$$

- We now compute the standard deviation σ of the random variable X as follows

$$\sigma = \text{Square Root} [\sum (X - \mu)^2 P(X)]$$

$$= \text{Square Root} [(0 - 1.5)^2 * (1/8) + (1 - 1.5)^2 * (3/8) + (2 - 1.5)^2 * (3/8) + (3 - 1.5)^2 * (1/8)]$$

Basic Probability Word Problems Practice Questions

Practice 1

Problems

1. Charles is spinning a spinner like the one shown in the figure while he is playing a game.

What is the probability that he spins a 5 on his next turn?

2. The letters from the word GEOMETRY are placed in a hat. What is the probability of selecting an M when you are choosing one letter without looking?
3. What is the probability of getting a 7 when you are rolling a number cube?
4. If the probability that it will rain is $\frac{1}{3}$, then what is the probability that it will not rain?
5. When you are selecting a number at random from the digits 0-9, what is the probability of **not** selecting a 3?

Solutions

1. *Read and understand the question.* This question is looking for the probability of getting a 5 when you are spinning a spinner. There are 6 equal sections to the spinner.

Make a plan. Use the fact that there are 6 equal sections and 1 of the sections is labeled with a 5.

Substitute into the formula

$$P(E) = \frac{\text{number of ways an event can occur}}{\text{total number of possible outcomes}}$$

Carry out the plan. The formula becomes $P(5) = \frac{1}{6}$. The probability is equal to $\frac{1}{6}$.

Check your answer. To check this answer, substitute the values again to see if the answer is reasonable. Since 1 out of the 6 sections is labeled with a 5, the probability is equal to $\frac{1}{6}$. This answer is checking.

2. *Read and understand the question.* This question is looking for the probability of selecting a letter M from the word GEOMETRY.

Make a plan. Use the facts that there are 8 letters in the word and 1 is a letter *M*. Substitute into the formula

$$P(E) = \frac{\text{number of ways an event can occur}}{\text{total number of possible outcomes}}$$

Carry out the plan. The formula becomes $P(M) = \frac{1}{8}$. The probability is equal to $\frac{1}{8}$.

Check your answer. To check this answer, substitute the values again to see if the answer is reasonable. Since 1 out of the 8 letters in the word is the letter *M*, the probability is equal to $\frac{1}{8}$. This answer is checking.

3. *Read and understand the question.* This question is looking for the probability of rolling a 7 on a number cube. There are 6 sides on a number cube.

Make a plan. Use the fact that there are 6 sides on the number cube and none of the sides have 7 dots. Substitute into the formula

$$P(E) = \frac{\text{number of ways an event can occur}}{\text{total number of possible outcomes}}$$

Carry out the plan. The formula becomes $P(7) = \frac{0}{6} = 0$. The probability is equal to 0.

Check your answer. This was an example of an impossible event. Thus, the probability is equal to 0. This answer is checking.

4. *Read and understand the question.* This question is looking for the probability that it will not rain when the probability that it will rain is given.

Make a plan. Use the fact that the probability that it will rain is $\frac{1}{3}$. Substitute into the formula $P(\text{not } E) = 1 - P(E)$.

Carry out the plan. The formula becomes $P(\text{not rain}) = 1 - \frac{1}{3} = \frac{3}{3} - \frac{1}{3} = \frac{2}{3}$. The probability is equal to $\frac{2}{3}$.

Check your answer. To check this answer, add the probability that it will rain to the probability that it will not rain. Make sure that the sum is equal to 1: $\frac{1}{3} + \frac{2}{3} = \frac{3}{3} = 1$. This answer is checking.

5. *Read and understand the question.* This question is looking for the probability of not selecting a 3 from the digits 0 - 9. There are 10 digits from 0 to 9.

Make a plan. Use the fact that the probability of selecting a 3 is $\frac{1}{10}$. Substitute into the formula $P(\text{not } E) = 1 - P(E)$.

Carry out the plan. The formula becomes $P(\text{not } 3) = 1 - \frac{1}{10} = \frac{10}{10} - \frac{1}{10} = \frac{9}{10}$. The probability is equal to $\frac{9}{10}$.

Check your answer. To check this answer, add the probability of selecting a 3 with the probability of not selecting a 3, and make sure that the sum is equal to 1: $\frac{1}{10} + \frac{9}{10} = \frac{10}{10} = 1$. This answer is checking.

Practice 2

Problems

1. What is the probability of selecting a red or a blue marble from a jar with 3 red marbles, 4 green marbles, and 2 blue marbles?
2. What is the probability of selecting a *P* or a vowel from the word PROBABILITY?
3. What is the probability of selecting an *I* or any vowel from the word PROBABILITY?
4. When you are selecting a card at random from a standard deck of 52 cards, what is the probability of selecting a king or a red card?

Solutions

1. *Read and understand the question.* This question is looking for the probability of selecting a red or a blue marble when you are picking a marble without looking. There are 3 red marbles, 4 green marbles, and 2 blue marbles in the jar.

Make a plan. Use the fact that there are a total of 9 marbles in the jar; 3 of them are red and 2 of them are blue. The key word *or* in the question tells you to add the probabilities together, and in this problem a marble cannot be red and blue at the same time, so the events are mutually exclusive.

Use the basic formula
$$P(E) = \frac{\text{number of ways an event can occur}}{\text{total number of possible outcomes}}$$
 and add the probabilities together.

Carry out the plan. The formula becomes $P(\text{red or blue}) = \frac{3}{9} + \frac{2}{9} = \frac{5}{9}$. The probability is equal to $\frac{5}{9}$.

Check your answer. To check this answer, substitute the values again to see if the answer is

reasonable. Since 5 out of the 9 marbles are either red or blue, this is equal to the probability of $\frac{5}{9}$. This answer is checking.

2. *Read and understand the question.* This question is looking for the probability of selecting a *P* or a vowel from the letters of the word PROBABILITY.

There are a total of 11 letters, one *P*, and 4 vowels.

Make a plan. Use the fact that there are a total of 11 letters in the word; 1 of them is a *P* and 4 of them are vowels. The key word *or* in the question tells you to add the probabilities together. *P* is not also a vowel, so the events are mutually exclusive. Use the basic

formula
$$P(E) = \frac{\text{number of ways an event can occur}}{\text{total number of possible outcomes}}$$
 and add the probabilities together.

Carry out the plan. The formula becomes $P(P \text{ or a vowel}) = \frac{1}{11} + \frac{4}{11} = \frac{5}{11}$

The probability is equal to $\frac{5}{11}$.

Check your answer. To check this answer, substitute the values again to see if the answer is reasonable. Since 5 out of the 11 letters are either the letter *P* or a vowel, this is equal to the probability of $\frac{5}{11}$. This answer is checking.

3. *Read and understand the question.* This question is looking for the probability of selecting an *I* or any vowel from the letters of the word PROBABILITY. There are a total of 11 letters, 2 *I*s, and 4 vowels, including the *I*s.

Make a plan. Use the fact that there are a total of 11 letters in the word; 2 of them are *I*s and a total of 4 are vowels. The key word *or* in the question tells you to add the probabilities together. *I* is also a vowel, so the events are *not* mutually exclusive. Use the basic

formula
$$P(E) = \frac{\text{number of ways an event can occur}}{\text{total number of possible outcomes}}$$
 and add the probabilities together, but then subtract the probability of the two *I*s because they are also vowels.

Carry out the plan. The formula becomes $P(I \text{ or any vowel}) = \frac{2}{11} + \frac{4}{11} - \frac{2}{11} = \frac{4}{11}$

The probability is equal to $\frac{4}{11}$.

Check your answer. To check this answer, substitute the values again to see if the answer is reasonable. Since 4 out of the 11 letters are vowels including the letter *I*, this is equal the probability of $\frac{4}{11}$. This answer is checking.

4. *Read and understand the question.* This question is looking for the probability of selecting a king or a red card when you are choosing from a deck of 52 cards. There are 26 red cards and 4 kings in the deck.

Make a plan. Use the facts that there are a total of 52 cards in the deck and 26 of them are red and 4 of them are kings. The key word or in the question tells you to add the probabilities together. In this problem, there are two cards that are red and a king at same time, so the events are not mutually

exclusive. Use the basic formula $P(E) = \frac{\text{number of ways an event can occur}}{\text{total number of possible outcomes}}$ and add the probabilities together, but subtract the probability of selecting a king that is also a red card.

Carry out the plan. The formula becomes $P(\text{red card or king}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$

The probability is equal to $\frac{7}{13}$.

Check your answer. To check this answer, substitute the values again to see if the answer is reasonable. Since 28 out of the 52 cards are either red or kings, this is equal to the probability of $\frac{28}{52} = \frac{7}{13}$. This answer is checking

Example 1

Suppose we draw a card from a deck of playing cards. What is the probability that we draw a spade?

Solution: The sample space of this experiment consists of 52 cards, and the probability of each sample point is $1/52$. Since there are 13 spades in the deck, the probability of drawing a spade is

$$P(\text{Spade}) = (13)(1/52) = 1/4$$

Example 2

Suppose a coin is flipped 3 times. What is the probability of getting two tails and one head?

Solution: For this experiment, the sample space consists of 8 sample points.

$$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$$

Each sample point is equally likely to occur, so the probability of getting any particular sample point is $1/8$. The event "getting two tails and one head" consists of the following subset of the sample space.

$$A = \{TTH, THT, HTT\}$$

The probability of Event A is the sum of the probabilities of the sample points in A. Therefore,

$$P(A) = 1/8 + 1/8 + 1/8 = 3/8$$

Example:

A survey was taken on 30 classes at a school to find the total number of left-handed students in each class. The table below shows the results:

No. of left-handed students	0	1	2	3	4	5
Frequency (no. of classes)	1	2	5	12	8	2

A class was selected at random.

- Find the probability that the class has 2 left-handed students.
- What is the probability that the class has at least 3 left-handed students?
- Given that the total number of students in the 30 classes is 960, find the probability that a student randomly chosen from these 30 classes is left-handed.

Solution:

a) Let S be the sample space and

A be the event of a class having 2 left-handed students.

$$n(S) = 30$$

$$n(A) = 5$$

$$P(A) = \frac{5}{30} = \frac{1}{6}$$

b) Let B be the event of a class having at least 3 left-handed students.

$$n(B) = 12 + 8 + 2 = 22$$

$$P(B) = \frac{22}{30} = \frac{11}{15}$$

c) First find the total number of left-handed students:

No. of left-handed students, x	0	1	2	3	4	5
Frequency, f (no. of classes)	1	2	5	12	8	2
fx	0	2	10	36	32	10

Total no. of left-handed students = $2 + 10 + 36 + 32 + 10 = 90$

Here, the sample space is the total number of students in the 30 classes, which was given as 960.

Let T be the sample space and C be the event that a student is left-handed.

$$n(T) = 960$$

$$n(C) = 90$$

$$P(C) = \frac{90}{960} = \frac{3}{32}$$

Before discussing the rules of probability, we state the following definitions:

- Two [events](#) are **mutually exclusive** or **disjoint** if they cannot occur at the same time.
- The probability that Event A occurs, given that Event B has occurred, is called a **conditional probability**. The conditional probability of Event A, given Event B, is denoted by the symbol $P(A|B)$.
- The **complement** of an event is the event not occurring. The probability that Event A will not occur is denoted by $P(A')$.
- The probability that Events A and B *both* occur is the probability of the **intersection** of A and B. The probability of the intersection of Events A and B is denoted by $P(A \cap B)$. If Events A and B are mutually exclusive, $P(A \cap B) = 0$.
- The probability that Events A or B occur is the probability of the **union** of A and B. The probability of the union of Events A and B is denoted by $P(A \cup B)$.
- If the occurrence of Event A changes the probability of Event B, then Events A and B are **dependent**. On the other hand, if the occurrence of Event A does not change the probability of Event B, then Events A and B are **independent**.

Probability Calculator

Use the Probability Calculator to compute the probability of an event from the known probabilities of other events. The Probability Calculator is free and easy to use. It can be found under the Stat Tools tab, which appears in the header of every Stat Trek web page.

[Probability Calculator](#)

Rule of Subtraction

In a [previous lesson](#), we learned two important properties of probability:

- The probability of an event ranges from 0 to 1.
- The sum of probabilities of all possible events equals 1.

The rule of subtraction follows directly from these properties.

Rule of Subtraction The probability that event A will occur is equal to 1 minus the probability that event A will not occur.

$$P(A) = 1 - P(A')$$

Suppose, for example, the probability that Bill will graduate from college is 0.80. What is the probability that Bill will not graduate from college? Based on the rule of subtraction, the probability that Bill will not graduate is $1.00 - 0.80$ or 0.20 .

Rule of Multiplication

The rule of multiplication applies to the situation when we want to know the probability of the intersection of two events; that is, we want to know the probability that two events (Event A and Event B) both occur.

Rule of Multiplication The probability that Events A and B both occur is equal to the probability that Event A occurs times the probability that Event B occurs, given that A has occurred.

$$P(A \cap B) = P(A) P(B|A)$$

Example

An urn contains 6 red marbles and 4 black marbles. Two marbles are drawn *without replacement* from the urn. What is the probability that both of the marbles are black?

Solution: Let A = the event that the first marble is black; and let B = the event that the second marble is black. We know the following:

- In the beginning, there are 10 marbles in the urn, 4 of which are black. Therefore, $P(A) = 4/10$.
- After the first selection, there are 9 marbles in the urn, 3 of which are black. Therefore, $P(B|A) = 3/9$.

Therefore, based on the rule of multiplication:

$$P(A \cap B) = P(A) P(B|A)$$

$$P(A \cap B) = (4/10) * (3/9) = 12/90 = 2/15$$

Rule of Addition

The rule of addition applies to the following situation. We have two events, and we want to know the probability that either event occurs.

Rule of Addition The probability that Event A or Event B occurs is equal to the probability that Event A occurs plus the probability that Event B occurs minus the probability that both Events A and B occur.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note: Invoking the fact that $P(A \cap B) = P(A)P(B|A)$, the Addition Rule can also be expressed as

$$P(A \cup B) = P(A) + P(B) - P(A)P(B|A)$$

Example

A student goes to the library. The probability that she checks out (a) a work of fiction is 0.40, (b) a work of non-fiction is 0.30, and (c) both fiction and non-fiction is 0.20. What is the probability that the student checks out a work of fiction, non-fiction, or both?

Solution: Let F = the event that the student checks out fiction; and let N = the event that the student checks out non-fiction. Then, based on the rule of addition:

$$\begin{aligned}P(F \cup N) &= P(F) + P(N) - P(F \cap N) \\P(F \cup N) &= 0.40 + 0.30 - 0.20 = 0.50\end{aligned}$$

Test Your Understanding of This Lesson

Problem 1

An urn contains 6 red marbles and 4 black marbles. Two marbles are drawn *with replacement* from the urn. What is the probability that both of the marbles are black?

- (A) 0.16
- (B) 0.32
- (C) 0.36
- (D) 0.40
- (E) 0.60

Solution

The correct answer is A. Let A = the event that the first marble is black; and let B = the event that the second marble is black. We know the following:

- In the beginning, there are 10 marbles in the urn, 4 of which are black. Therefore, $P(A) = 4/10$.
- After the first selection, we replace the selected marble; so there are still 10 marbles in the urn, 4 of which are black. Therefore, $P(B|A) = 4/10$.

Therefore, based on the rule of multiplication:

$$\begin{aligned}P(A \cap B) &= P(A) P(B|A) \\P(A \cap B) &= (4/10) * (4/10) = 16/100 = 0.16\end{aligned}$$

Problem 2

A card is drawn randomly from a deck of ordinary playing cards. You win \$10 if the card is a spade or an ace. What is the probability that you will win the game?

- (A) 1/13
- (B) 13/52

- (C) 4/13
- (D) 17/52
- (E) None of the above.

Solution

The correct answer is C. Let S = the event that the card is a spade; and let A = the event that the card is an ace. We know the following:

- There are 52 cards in the deck.
- There are 13 spades, so $P(S) = 13/52$.
- There are 4 aces, so $P(A) = 4/52$.
- There is 1 ace that is also a spade, so $P(S \cap A) = 1/52$.

Therefore, based on the rule of addition:

$$P(S \cup A) = P(S) + P(A) - P(S \cap A)$$
$$P(S \cup A) = 13/52 + 4/52 - 1/52 = 16/52 = 4/13$$

Which of the following is a discrete random variable?

- I. The average height of a randomly selected group of boys.
- II. The annual number of sweepstakes winners from New York City.
- III. The number of presidential elections in the 20th century.

- (A) I only
- (B) II only
- (C) III only
- (D) I and II
- (E) II and III

Solution

The correct answer is B.

The annual number of sweepstakes winners results from a random process, but it can only be a whole number - not a fraction; so it is a discrete random variable. The average height of a randomly-selected group of boys could take on any value between the height of the smallest and tallest boys, so it is not a discrete variable. And the number of presidential elections in the 20th century does not result from a random process; so it is not a random variable

Problem 1

The number of adults living in homes on a randomly selected city block is described by the following probability distribution.

Number of adults, x	1	2	3	4 or more
Probability, P(x)	0.25	0.50	0.15	???

What is the probability that 4 or more adults reside at a randomly selected home?

- (A) 0.10
- (B) 0.15
- (C) 0.25
- (D) 0.50
- (E) 0.90

Solution

The correct answer is A. The sum of all the probabilities is equal to 1. Therefore, the probability that four or more adults reside in a home is equal to $1 - (0.25 + 0.50 + 0.15)$ or 0.10.

Mean and Variance of Random Variables



Just like [variables](#) from a data set, [random variables](#) are described by measures of central tendency (like the mean) and measures of variability (like variance). This lesson shows how to compute these measures for [discrete](#) random variables.

Mean of a Discrete Random Variable

The mean of the discrete random variable X is also called the **expected value** of X . Notationally, the expected value of X is denoted by $E(X)$. Use the following formula to compute the mean of a discrete random variable.

$$E(X) = \mu_x = \sum [x_i * P(x_i)]$$

where x_i is the value of the random variable for outcome i , μ_x is the mean of random variable X , and $P(x_i)$ is the probability that the random variable will be outcome i .

Example 1

In a recent little league softball game, each player went to bat 4 times. The number of hits made by each player is described by the following probability distribution.

Number of hits, x	0	1	2	3	4
Probability, P(x)	0.10	0.20	0.30	0.25	0.15

What is the mean of the probability distribution?

- (A) 1.00
- (B) 1.75
- (C) 2.00
- (D) 2.25
- (E) None of the above.

Solution

The correct answer is E. The mean of the probability distribution is 2.15, as defined by the following equation.

$$E(X) = \sum [x_i * P(x_i)]$$
$$E(X) = 0*0.10 + 1*0.20 + 2*0.30 + 3*0.25 + 4*0.15 = 2.15$$

Median of a Discrete Random Variable

The median of a discrete random variable is the "middle" value. It is the value of X for which $P(X \leq x)$ is greater than or equal to 0.5 and $P(X \geq x)$ is greater than or equal to 0.5.

Consider the problem presented above in Example 1. In Example 1, the median is 2; because $P(X \leq 2)$ is equal to 0.60, and $P(X \geq 2)$ is equal to 0.70. The computations are shown below.

$$P(X \leq 2) = P(x=0) + P(x=1) + P(x=2) = 0.10 + 0.20 + 0.30 = 0.60$$

$$P(X \geq 2) = P(x=2) + P(x=3) + P(x=4) = 0.30 + 0.25 + 0.15 = 0.70$$

Variability of a Discrete Random Variable

The equation for computing the variance of a discrete random variable is shown below.

$$\sigma^2 = \sum \{ [x_i - E(x)]^2 * P(x_i) \}$$

where x_i is the value of the random variable for outcome i , $P(x_i)$ is the probability that the random variable will be outcome i , $E(x)$ is the expected value of the discrete random variable x .

Example 2

The number of adults living in homes on a randomly selected city block is described by the following probability distribution.

Number of adults, x	1	2	3	4
Probability, P(x)	0.25	0.50	0.15	0.10

What is the standard deviation of the probability distribution?

- (A) 0.50
- (B) 0.62
- (C) 0.79

(D) 0.89

(E) 2.10

Solution

The correct answer is D. The solution has three parts. First, find the expected value; then, find the variance; then, find the standard deviation. Computations are shown below, beginning with the expected value.

$$E(X) = \sum [x_i * P(x_i)]$$
$$E(X) = 1*0.25 + 2*0.50 + 3*0.15 + 4*0.10 = 2.10$$

Now that we know the expected value, we find the variance.

$$\sigma^2 = \sum \{ [x_i - E(x)]^2 * P(x_i) \}$$
$$\sigma^2 = (1 - 2.1)^2 * 0.25 + (2 - 2.1)^2 * 0.50 + (3 - 2.1)^2 * 0.15 + (4 - 2.1)^2 * 0.10$$
$$\sigma^2 = (1.21 * 0.25) + (0.01 * 0.50) + (0.81) * 0.15 + (3.61 * 0.10) = 0.3025 + 0.0050 + 0.1215 + 0.3610 = 0.79$$

And finally, the standard deviation is equal to the square root of the variance; so the standard deviation is $\sqrt{0.79}$ or 0.889.

Independent Random Variables

View Video

When a study involves pairs of [random variables](#), it is often useful to know whether or not the random variables are independent. This lesson explains how to assess the independence of random variables.

Independence of Random Variables

If two random variables, X and Y, are **independent**, they satisfy the following conditions.

- $P(x|y) = P(x)$, for all values of X and Y.
- $P(x \cap y) = P(x) * P(y)$, for all values of X and Y.

The above conditions are equivalent. If either one is met, the other condition also met; and X and Y are independent. If either condition is not met, X and Y are **dependent**.

Note: If X and Y are independent, then the [correlation](#) between X and Y is equal to zero.

Joint Probability Distributions

	X		
0	0	1	2

	3	0.1	0.2	0.2
Y	4	0.1	0.2	0.2

The table on the right shows the joint probability distribution between two discrete random variables - X and Y.

In a joint probability distribution table, numbers in the cells of the table represent the probability that particular values of X and Y occur together. From this table, you can see that the probability that X=0 and Y=3 is 0.1; the probability that X=1 and Y=3 is 0.2; and so on.

You can use tables like this to figure out whether two discrete random variables are independent or dependent. Problem 1 below shows how.

Test Your Understanding of This Lesson

Problem 1

The table on the left shows the joint probability distribution between two random variables - X and Y; and the table on the right shows the joint probability distribution between two random variables - A and B.

		X		
		0	1	2
	3	0.1	0.2	0.2
Y	4	0.1	0.2	0.2

		A		
		0	1	2
	3	0.1	0.2	0.2
B	4	0.2	0.2	0.1

Which of the following statements are true?

- I. X and Y are independent random variables.
- II. A and B are independent random variables.

- (A) I only
- (B) II only
- (C) I and II
- (D) Neither statement is true.
- (E) It is not possible to answer this question, based on the information given.

Solution

The correct answer is A. The solution requires several computations to test the independence of random variables. Those computations are shown below.

X and Y are independent if $P(x|y) = P(x)$, for all values of X and Y. From the probability distribution table, we know the following:

$$\begin{array}{lll}
P(x=0) = 0.2; & P(x=0 \mid y=3) = 0.2; & P(x=0 \mid y = 4) = 0.2 \\
P(x=1) = 0.4; & P(x=1 \mid y=3) = 0.4; & P(x=1 \mid y = 4) = 0.4 \\
P(x=2) = 0.4; & P(x=2 \mid y=3) = 0.4; & P(x=2 \mid y = 4) = 0.4
\end{array}$$

Thus, $P(x|y) = P(x)$, for all values of X and Y, which means that X and Y are independent. We repeat the same analysis to test the independence of A and B.

$$\begin{array}{lll}
P(a=0) = 0.3; & P(a=0 \mid b=3) = 0.2; & P(a=0 \mid b = 4) = 0.4 \\
P(a=1) = 0.4; & P(a=1 \mid b=3) = 0.4; & P(a=1 \mid b = 4) = 0.4 \\
P(a=2) = 0.3; & P(a=2 \mid b=3) = 0.4; & P(a=2 \mid b = 4) = 0.2
\end{array}$$

Thus, $P(a|b)$ is not equal to $P(a)$, for all values of A and B. For example, $P(a=0) = 0.3$; but $P(a=0 \mid b=3) = 0.2$. This means that A and B are *not* independent.