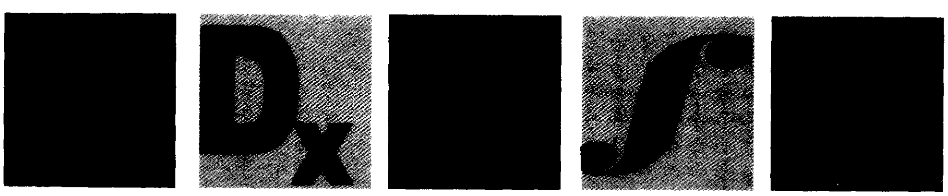
Differential & Integral Calculus

FELICIANO and UY



n general, if there is a relation between two variables x and y such hat for each value of x, there corresponds a value of y, then y is aid to be a function of x. Symbolically, this is written in the orm*

$$y = f(x)$$

The function concept may be extended to relations between ore than two variables. Consider the equation

$$z = f(x,y).$$

his implies that z is determined when x and y are given and it is istomary to say that z is a function of x and y. For instance, the olume of a right circular cylinder is a function of the altitude h id radius r of the base. that is,

$$V = f(r,h) = \pi r^2 h.$$

It is important that we be familiar with the *functional notaon* In mathematics and the physical sciences functional notion plays a convenient and important part. In the example elow, we shall illustrate how to set up a formula showing the inctional relation between the variables.

XAMPLE: The area of a rectangle is 6 sq. in. Express the perimeter P of the rectangle as a function of the length x of one side.

OLUTION: Since the area is 6 sq. in., then the length of the other side is $\frac{6}{x}$ and the perimeter is

$$P = 2\left(x + \frac{6}{x}\right)$$

*The notation $y = f(x)$ is due to the Swiss mathematician Leonard Euler (1707 83).

EXERCISE 1.1

1. If $f(x) = x^2 - 4x$, find (a) $f(-5)$ (b) $f(y^2 + 1)$ (c) $f(x + \Delta x)$ (d) $f(x + 1) - f(x - 1)$.
2. If $y = \frac{x^2 + 3}{x}$, find x as a function of y.
3. If $y = \tan(x + \pi)$, find x as a function of y.
4. Express the distance D traveled in t hr by a car whose speed is 60 km/hr.
5. Express the area A of an equilateral triangle as a function of its side x.
6. The stiffness of a beam of rectangular cross section is proportional to the breadth and the cube of the depth. If the breadth is 20 cm, express the stiffness as a function of the depth.
7. A right circular cylinder, radius of base x, height y, is inscribed in a right circular cone, radius of base r and height h Express y as function of x (r and h are constants).
8. If $f(x) = x^2 + 1$, find $\frac{f(x+h) - f(x)}{h}$, $h \neq 0$.
9. If $f(x) = 3x^2 - 4x + 1$, find $\frac{f(h + 3) - f(3)}{h}$, $h \neq 0$.
10. If $f(x) = \frac{4}{x+3}$ and $g(x) = x^2 - 3$, find $f[g(x)]$ and $g[f(x)]$

1.2 Limit of a Function

Familiarity with the limit concept is absolutely essential for a deeper understanding of the calculus. In this section, we shall begin our discussion of the limit of a function but we emphasize that our treatment here will appeal more to our intuition than to rigor. And since our approach is a non-rigorous one, we therefore, expect you to grasp this idea with ease.

The example above illustrates the fact that f(x) may have a limit at a number a even though the value $f(a)$ of the function is undefined. Moreover, it shows that the limit and value of the function are two different concepts.

EXAMPLE 2. Evaluate $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ if $f(x) = x^2 - 3x$

Solution: A straight substitution of $x = 2$ leads to the indeterminate form $\frac{0}{0}$. Since $f(x) = x^2 - 3x$, then $f(2) = 4 - 6 = -2$. Hence.

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{(x^2 - 3x) - (-2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x - 1)(x - 2)}{x - 2}$$

$$= \lim_{x \to 2} (x - 1)$$

$$= 1$$

EXERCISE 1.3

Evaluate each of the following:

1. $\lim_{x \to 4} \frac{x^3 - 64}{x^2 - 16}$
2. $\lim_{x \to 2} \frac{x^2 + 2x - 8}{3x - 6}$
3. $\lim_{x \to 3} \frac{x^3 - 13x + 12}{x^3 - 14x + 15}$
4. $\lim_{x \to 2} \frac{x^3 - x^2 - x - 2}{2x^3 - 5x^2 + 5x - 6}$

5. $\lim_{x \to 0} \frac{(x + 3)^2 - 9}{2x}$
6. $\lim_{x \to 0} \frac{\sqrt{x + 16} - 4}{x}$
7. $\lim_{x \to 1} \frac{x - 1}{\sqrt{x + 3} - 2}$
8. $\lim_{x \to 8} \frac{\sqrt[3]{x} - 2}{x - 8}$
9. $\lim_{x \to 4} \frac{\frac{1}{x} - \frac{1}{4}}{x - 4}$
10. $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$
11. $\lim_{x \to 3} \frac{x - 3}{\sqrt{x - 2} - \sqrt{4 - x}}$
12. $\lim_{x \to 0} \frac{1}{x}\left(\frac{1}{3} - \frac{1}{\sqrt{x + 9}}\right)$
13. $\lim_{x \to 3} \frac{\sqrt{x^2 - 9}}{x - 3}$
14. $\lim_{x \to \frac{\pi}{4}} \frac{\tan 2x}{\sec 2x}$
15. $\lim_{x \to 0} \frac{\sin^3 x}{\sin x - \tan x}$
16. $\lim_{x \to 0} \frac{1 - \cos^2 x}{1 + \cos x}$
17. $\lim_{x \to 0} \frac{\sin x \sin 2x}{1 - \cos x}$

18. $\lim_{x \to \pi} \dfrac{\sin^2 x}{1 + \cos x}$

If $f(x) = \sqrt{x}$, find

19. $\lim_{x \to 4} \dfrac{f(x) - f(4)}{x - 4}$

20. $\lim_{x \to 0} \dfrac{f(9 + x) - f(9)}{x}$

If $f(x) = x^2 - 2x + 3$, find

21. $\lim_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$

22. $\lim_{x \to 0} \dfrac{f(x + 2) - f(2)}{x}$

1.5 Infinity

Let $f(x)$ be a function. If we can make $f(x)$ as large as we please by making x close enough, but not equal, to a real number a, then we describe this situation by writing.

$$\lim_{x \to a} f(x) = \infty$$

where the symbol ∞ is read "infinity".

In particular, consider the function $f(x) = \dfrac{1}{x}$. The table below shows that as x takes on values successively approaching the number 0, the value of $\dfrac{1}{x}$ grows larger and larger. We say that $\dfrac{1}{x}$ becomes infinite as x approaches 0 and indicate this by writing

$$\frac{1}{x} \to \infty \quad \text{as} \quad x \to 0$$

In more compact form, we write

$$\lim_{x \to 0} \frac{1}{x} = \infty$$

x	0.1000	0.0100	0.0010	0.0001	$\to 0$
$f(x) = \dfrac{1}{x}$	10	100	1,000	10,000	$\to \infty$

Bear in mind that ∞ is not a number which results from division by zero. Recall that in the real number system, division by zero is not permissible. In fact, it can be argued that the statement

$$\lim_{x \to a} f(x) = \infty$$

is not an equation at all since ∞ does not represent a number. It is merely used as a symbol to imply that the value of $f(x)$ increases numerically without bound as x approaches a.*

1.6 Limit at Infinity

A function $f(x)$ may have a finite limit even when the independent variable x becomes infinite. This statement "x becomes infinite" is customarily expressed in symbolism by "$x \to \infty$".

Consider again the function $f(x) = \dfrac{1}{x}$. It can be shown (intuitively or formally) that $\dfrac{1}{x}$ approaches a finite limit (the number zero) as x increases without bound. That is,

*The symbol ∞ is used for infinity if no particular reference to sign is made. The symbols $+\infty$ (read "plus infinity") and $-\infty$ (read "minus infinity") are used in some books in connection with statements about limits. The symbol $+\infty$ is used to indicate that $f(x)$ becomes positively infinite (increases without bound) while $-\infty$ is used to mean that $f(x)$ becomes negatively infinite (decreases without bound).

$$\frac{1}{x} \to 0 \text{ as } x \to \infty$$

We shall consider this fact as an additional theorem on limits a in symbol, we write

$$\text{L9} \qquad \lim_{x\to\infty} \frac{1}{x} = 0$$

The use of L9 is illustrated in the following examples.

EXAMPLE 1. $\lim\limits_{x\to\infty} \frac{1}{x^3} = \lim\limits_{x\to\infty} \left(\frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x}\right)$

$= \lim\limits_{x\to\infty} \frac{1}{x} \cdot \lim\limits_{x\to\infty} \frac{1}{x} \cdot \lim\limits_{x\to\infty} \frac{1}{x}$ by L

$= 0$

EXAMPLE 2. $\lim\limits_{x\to\infty} \frac{4}{x^2} = 4 \lim\limits_{x\to\infty} \frac{1}{x^2}$ why

$= 4 \lim\limits_{x\to\infty} \left(\frac{1}{x} \cdot \frac{1}{x}\right)$

$= 4 \lim\limits_{x\to\infty} \frac{1}{x} \cdot \lim\limits_{x\to\infty} \frac{1}{x}$ by L

$= 0$ by L

EXAMPLE 3. $\lim\limits_{x\to\infty} \frac{1}{x^{\frac{1}{4}}} = \lim\limits_{x\to\infty} \left(\frac{1}{x}\right)^{\frac{1}{4}}$ why

$= \left[\lim\limits_{x\to\infty} \frac{1}{x}\right]^{\frac{1}{4}}$ by L8

$= 0$ by L9

From the examples above, we intuitively feel that if n is any positive number, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0$$

This is given as a theorem in some books. Note that when n= 1, we have L9.

A function $f(x) = \frac{N(x)}{D(x)}$ may assume the indeterminate form $\frac{\infty}{\infty}$ when x is replaced by ∞. However, the limit of f(x) as x becomes infinite may be definite. To find this limit we first divide N(x) and D(x) by the highest power of x. Then we evaluate the limit by use of L9.

EXAMPLE: Evaluate $\lim\limits_{x\to\infty} \frac{4x^3 + 3x^2 - 6}{2x^3 + 5x + 3}$

Solution: The function assumes the indeterminate form $\frac{\infty}{\infty}$ when x is replaced by ∞. Dividing the numerator and denominator by x^3, we get

$$\lim_{x\to\infty} \frac{4x^3 + 3x^2 - 6}{2x^3 + 5x + 3} = \lim_{x\to\infty} \frac{4 + \frac{3}{x} - \frac{6}{x^3}}{2 + \frac{5}{x^2} + \frac{3}{x^3}}$$

$$= \frac{5 + 0 - 0}{2 + 0 + 0}$$

$$= 2$$

EXERCISE 1.4

Evaluate each of the following.

1. $\lim\limits_{x\to\infty} \frac{6x^3 + 4x^2 + 5}{8x^3 + 7x - 3}$

5. $\lim\limits_{x\to\infty} \frac{8x - 5}{\sqrt{4x^2 + 3}}$

2. $\lim_{x \to \infty} \frac{3x^2 + x + 2}{x^3 + 8x + 1}$

3. $\lim_{x \to \infty} \frac{4x + 5}{x^2 + 1}$

4. $\lim_{x \to \infty} \frac{x^3 + x + 2}{x^2 - 1}$

6. $\lim_{x \to \infty} \frac{x^3}{(2x - 1)^2}$

7. $\lim_{x \to \infty} \frac{(x + 2)^3 - (x - 2)^3}{x^2}$

8. $\lim_{x \to \infty} \frac{\sqrt{9x^2 + 4}}{6x + 1}$

1.7 Continuity

In Section 1.4, we emphasized that the *limit* and *value* of a function are two different concepts. In fact, in Section 1.2, when we discussed the meaning of $\lim_{x \to a} f(x) = L$, we deliberately ignored the actual value of f(x) at $x = a$. However, in Section 1.3, we made mention of the fact that the limit of a function f(x) as $x \to a$ may turn out to be just the value of f(x) at $x = a$. That is, $\lim_{x \to a} f(x) = f(a)$. Now when this happens, we have an event of some mathematical significance. The function f(x) is said to be *continuous* at $x = a$. This leads to the following definition.

DEFINITION 1.2 A function f(x) is continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

Note that the condition $\lim_{x \to a} f(x) = f(a)$ in the definition* above actually implies three conditions, namely

(1) $f(a)$ is defined.

(2) $\lim_{x \to a} f(x) = L$ exists, and

(3) $L = f(a)$

If any of these conditions is not satisfied, then f(x) is said to be *discontinuous* at $x = a$.

A function f(x) is said to be *continuous in an interval if it is continuous for every value of x in the interval.* The graph of

*This definition was formulated by the French mathematician Augustin Louis Cauchy (1789-1857).

this function is "unbroken" over that interval. That is, the graph of f(x) can be drawn without lifting the pencil from the paper (see Fig. 1.1).

EXAMPLE 1. The function $f(x) = x^2$ is continuous at $x = 2$ because $\lim_{x \to 2} x^2 = f(2) = 4$. In fact, it is continuous for all finite values of x. The graph of the function is shown in Fig. 1.1.

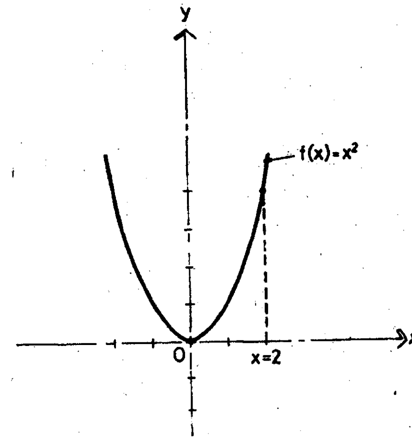


FIG. 1.1

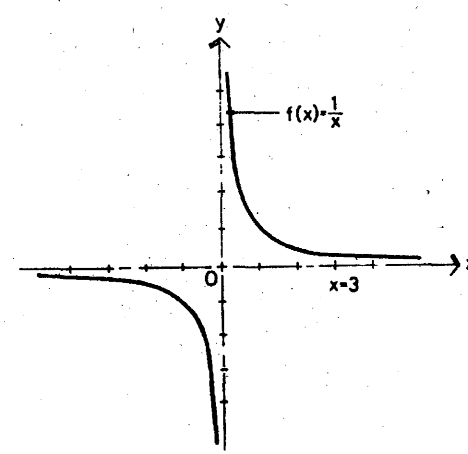


FIG. 1.2

EXAMPLE 2. The function $f(x) = \frac{1}{x}$ is continuous at $x = 3$ because $\lim_{x \to 3} \frac{1}{x} = f(3) = \frac{1}{3}$. It is, however, discontinuous at $x = 0$ since $\lim_{x \to 0} \frac{1}{x} = \infty$. The graph of the function (see Fig. 1.2) contains a "break" at $x = 0$.

EXAMPLE 3. Is the function $f(x) = \frac{4x}{x^2 - 4}$ continuous over the interval $0 \le x \le 5$?

Answer: No, since at $x = 2$, $f(2)$ is undefined.

EXERCISE 1.5

Find the value or values of x for which the function is discontinuous.

1. $\frac{3x}{x-5}$
2. $\frac{3x+2}{x^2 - 8x + 15}$
3. $\frac{5x+1}{x^2+4}$
4. $\frac{6x}{x^2-9}$
5. $\frac{1}{2^x - 8}$
6. $\frac{x+3}{x^3 - 3x^2 + 2x}$

1.8 Asymptotes

Let $f(x) = \frac{N(x)}{D(x)}$, $D(x) \neq 0$, be a rational function, i.e. N(x) and D(x) are polynomials. Suppose we wish to sketch the graph of f(x). A useful aid in sketching the graph of a function is to find, if there is any, the *asymptote* of its graph. The asymptote may be a *vertical line* (no slope), a *horizontal line* (zero slope) or a *non-vertical line* which slants upward to the right (positive slope) or slants downward to the right (negative slope). The following definitions are used to determine the vertical and horizontal asymptotes.

DEFINITION 1.3 The line $x = a$ is a vertical asymptote of the graph of f(x) if $\lim_{x \to a} f(x) = \infty$.

DEFINITION 1.4 The line $y = b$ is a horizontal asymptote of the graph of f(x) if $\lim_{x \to \infty} f(x) = b$.

EXAMPLE 1. Since $\lim_{x \to 3} \frac{2x}{x-3} = \infty$, then $x = 3$ is a vertical asymptote of the graph of the function defined by $f(x) = \frac{2x}{x-3}$.

EXAMPLE 2. $y = 2$ is a horizontal asymptote of the graph of $f(x) = \frac{4x^2}{2x^2 - 6}$ since $\lim_{x \to \infty} \frac{4x^2}{2x^2 - 6} = 2$.

EXAMPLE 3. $y = 0$ is a horizontal asymptote of the of $f(x) = \frac{3x}{x^2 - 1}$ since $\lim_{x \to \infty} \frac{3x}{x^2 - 1} = 0$.

EXAMPLE 4. There is no horizontal asymptote for the graph of $f(x) = \frac{4x^2}{2x-1}$ since $\lim_{x \to \infty} \frac{4x^2}{2x-1} = \infty$.

From Definitions 1.3 and 1.4 and the examples above, we can make certain generalizations which would facilitate further the process of finding the vertical and horizontal asymptotes* of the graph of the rational function defined by the equation

$$f(x) = \frac{N(x)}{D(x)}, \quad D(x) \neq 0$$

Since N(x) and D(x) are polynomials, we may let

$$N(x) = a_0 x^m + a_1 x^{m-1} + \ldots + a_{m-1} + a_m$$

$$D(x) = b_0 x^n + b_1 x^{n-1} + \ldots + b_{n-1} + b_n$$

where m and n are positive integers and $a_0, a_1, \ldots a_m$ and $b_0, b_1, \ldots b_n$ are constants. We now formulate the following rules for

*Other properties of a curve such as its intercepts and symmetry are assumed familiar to the student.

$$= \lim_{\Delta t \to 0} \frac{[3(t+\Delta t)^2 - 4] - [3t^2 - 4]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{3(t+\Delta t)^2 - 3t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{6t \cdot \Delta t + 3(\Delta t)^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} (6t + \Delta t)$$

$$= 6t$$

EXERCISE 2.1

Find the derivative by use of Definition 2.1

1. $y = 4x^2 - 5x$
2. $y = x^3 + 2x$
3. $y = 4\sqrt{x}$
4. $y = \dfrac{6}{x}$
5. $y = \sqrt[3]{x}$
6. $y = 2 - 5x$
7. $y = \sqrt{4x+3}$
8. $y = \dfrac{2x}{x+1}$
9. $y = \dfrac{3}{\sqrt{2x+1}}$
10. $y = \dfrac{5x^2}{4x-1}$
11. Given $s = \sqrt{t} - 2$, find $\dfrac{ds}{dt}$
12. Given $A = \pi r^2$ find $\dfrac{dA}{dr}$

13. Given $V = \dfrac{4}{3}\pi r^3$, find $\dfrac{dV}{dr}$
14. Given $S = 4\pi r^2$, find $\dfrac{dS}{dr}$
15. Given $S = \dfrac{2t+3}{3t-4}$, find $\dfrac{dS}{dt}$

Geometric Significance of $\frac{dy}{dx}$

Consider the graph of $y = f(x)$ shown in Fig. 2.2. Let $P(x,y)$ $Q(x + \Delta x, y + \Delta y)$ be any two points on this curve. Line S intersects the curve at P and Q and having inclination α is the *secant line* of the curve. Note that the slope of S is

$$m = \tan\alpha = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

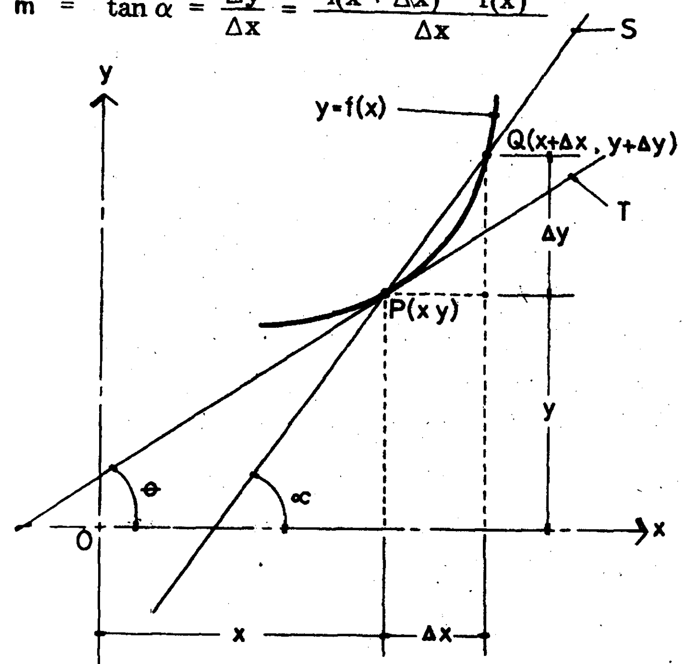


FIG. 2.2

$$= \frac{(2x+1)^3 (0) - 12 (2x+1)^2 (2)}{(2x+1)^6}$$

$$= \frac{-24 (2x+1)^2}{(2x+1)^6}$$

$$= \frac{-24}{(2x+1)^4}$$

2nd Solution: $y = \frac{4}{(2x+1)^3} = 4 (2x+1)^{-3}$ why?

$\frac{dy}{dx} = 4 \frac{d}{dx} (2x+1)^{-3}$ by D3

$dx \;\; = 4(-3) (2x+1)^{-4} \frac{d}{dx} (2x+1)$ by D7

$= -12 (2x+1)^{-4} (2)$

$= -24 (2x+1)^{-4}$

$= \frac{24}{(2x+1)^4}$

3rd Solution: $y = \frac{4}{(2x+1)^3}$

$\frac{dy}{dx} = \frac{4(-3)}{(2x+1)^4} \frac{d}{dx} (2x+1)$ by D8

$= \frac{-12}{(2x+1)^4} (2)$

$= \frac{-24}{(2x+1)^4}$

AMPLE 4. Find $\frac{dy}{dx}$ if $y = (2x+1)^3 (4x-1)^2$.

Solution:

$\frac{dy}{dx} = (2x+1)^3 \frac{d}{dx} (4x-1)^2 + (4x-1)^2 \frac{d}{dx} (2x+1)^3$ by D5

$(2x+1)^3 . 2 (4x-1) (4) + (4x-1)^2 . 3 (2x+1)^2 (2)$. by D7

$2(2x+1)^2 (4x-1) \left[4(2x+1) + 3(4x-1)\right]$

$2(2x+1)^2 (4x-1) (20x+1)$

EXERCISE 2.2

$\frac{dy}{dx}$ of each of the following:

1. $y = 5x^3 - 4x^2 + 3x - 6$
2. $y = \sqrt[3]{x} + \frac{4}{x} + \sqrt{x}$
3. $y = \sqrt{5 - 6x}$
4. $y = \sqrt[3]{2x-7}$
5. $y = (3x^2 - 4x + 1)^5$
6. $y = \sqrt{7 + \sqrt{3x+1}}$
7. $y = \frac{4x-5}{2x+1}$
8. $y = \frac{3x+1}{\sqrt{3x^2+2}}$
9. $y = (2x+5) \sqrt{4x-1}$
10. $y = (3x+4)^2 (x-5)^3$

11. $y = \left(\frac{2x-3}{5x+1}\right)^4$

12. $y = \sqrt{\frac{3x-4}{2x+5}}$

13. $y = \left(\frac{x-6}{3x+4}\right)^{\frac{1}{3}}$

14. $y = \sqrt[3]{x^2} - 4x^{-3}$

15. $y = 4(\sqrt{x} + 1)^5$

16. $y = \frac{4}{\sqrt{5x+3}}$

17. $y = \frac{2}{(4x+1)^3}$

Evaluate $\frac{dy}{dx}$ at the specified value of x:

18. $y = 6(\sqrt[3]{x} + 2)^2, \; x = 8$

19. $y = \sqrt{6 - \sqrt{x}} \;, \; x = 4$

20. $y = x^3 + 4x^{-1}, \; x = 1$

21. $y = (2x-1)^3 + \frac{4}{\sqrt{3x-2}} \qquad x = 2$

Find the slope of the tangent to the curve at the given point

22. $y = 7 - x^2 + 4x^3 \;, (-1, 2)$

23. $y = x + 2x^{-1} \;, (2,3)$

24. $y = 3x^2 - \frac{4}{x}, \; (2,10)$

25. $y = \frac{\sqrt{10-2x}}{3x}, \; (3, 2/9)$

Find the values of x for which the derivative is zero.

26. $y = x^3 + 4x^2 - 3x - 5$

27. $y = x^4 - 8x^3 + 22x^2 - 24x + 9$

28 $y'' = 12x + 8x^{-1}$

29. $y = \frac{x-1}{x^2 - 2x + 5}$

Find the values of x given that

30. $y = 2x - 3x^{-1}$ and $\frac{dy}{dx} = 14$

31. $y = x^{\frac{2}{3}} - x^{\frac{1}{3}}$ and $\frac{dy}{dx} = \frac{1}{4}$

32. $y = 3x^2 + 4x^{-1}$ and $\frac{dy}{dx} = 11$

2.5 The Chain Rule

Certain functions are formed out of simpler functions by a process of substitution. Functions which result in this manner are called *composite functions*.

For a general discussion of composite functions, consider the functions f and g given by $y = f(u)$ and $u = g(x)$ respectively. We have here a situation in which y depends on u and u in turn depends on x. To eliminate u, we simply substitute $u = g(x)$ in $y = f(u)$ and thereby obtain a new function h expressed symbolically in the form

The functions f and g are said to be *inverse functions*. To distinguish between f and g, we shall call f the *direct function* and g the *inverse function*.

Let us now focus our attention to the problem of finding the derivative of y with respect to x or $\frac{dy}{dx}$ of a function written in the form $x = g(y)$. This is a accomplished by using the so called *inverse function rule* which we state as follows:

INVERSE FUNCTION RULE:

If y is a differentiable function of x defined by $y = f(x)$, then its inverse function defined by $x = g(y)$ is a differentiable function of y and

D11: $$\frac{dy}{dx} = \frac{1}{dx/dy}$$

Note that D11 clearly shows that the rate of change of y with respect to x (dy/dx) and the rate of change of x with respect to y (dx/dy) are reciprocals. It also says that *the derivative of the inverse function is equal to the reciprocal of the derivative of the direct function*. The proof of D11 is given below.

Proof of D11: Let $y = f(x)$ and $x = g(y)$ be inverse functions. Then y is a function x and x is a function of y. By D10,

$$\frac{dy}{dy} = \frac{dy}{dx} \cdot \frac{dx}{dy}$$

and $$1 = \frac{dy}{dx} \cdot \frac{dx}{dy}$$

or $$\frac{dy}{dx} = \frac{1}{dx/dy}$$

EXAMPLE. If $x = y^3 - 4y^2$, , find $\frac{dy}{dx}$.

Solution: Since $x = y^3 - 4y^2$, then $\frac{dx}{dy} = 3y^2 - 8y$

and by D11,

$$\frac{dy}{dx} = \frac{1}{3y^2 - 8y}$$

EXERCISE 2.3

...the Chain Rule to find $\frac{dy}{dx}$ and express the final answer in ... of x.

1. $y = u^2 + u$, $u = 2x + 1$
2. $y = \sqrt{u^2 - 1}$, $u = 4\sqrt{x}$
3. $y = (u - 4)^{\frac{3}{2}}$, $u = x^2 + 4$
4. $y = (2u - 2)^{\frac{2}{3}}$, $u = 4x^3 + 1$
5. $y = \sqrt{u + 2}$, $u = 4x - 2$
6. $y = \dfrac{2u}{u^2 - 1}$, $u = x^2$
7. $y = \sqrt{u}$, $u = \sqrt{x}$

... Inverse Function Rule to find $\frac{dy}{dx}$.

8. $x = y + y^2 + y^3$
9. $x = \sqrt{y} + \sqrt[3]{y}$
10. $x = (4 - 3y)^{\frac{3}{2}}$
11. $x = 2(4y + 1)^3$
12. $x = \dfrac{6}{(3y+1)^2}$
13. $x = \sqrt{1 + \sqrt{1 + \sqrt{y}}}$
14. $x = \left(\dfrac{2y + 1}{3y - 1}\right)^4$

2.7 Higher Derivatives

Recall that from the equation $y = f(x)$, we get by diff tiation the equation

$$\frac{dy}{dx} = f'(x)$$

The derivative $\frac{dy}{dx}$ or $f'(x)$ of the function f is a number tha pends on x. Hence f' is itself a function of x and may be diff tiated again with respect to x. This process is represented sy lically by any of the following notations:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$$\frac{d}{dx}[f'(x)] = f''(x)$$

$$\frac{d}{dx}(y') = y''$$

$$D_x(D_x y) = D_x^2 y$$

If we refer to $\frac{dy}{dx}$ as the first derivative of $y = f(x)$, the shall refer to $\frac{d^2y}{dx^2}$ (read "d squared y dx squared") as the *se derivative* of $y = f(x)$. The operator $\frac{d^2}{dx^2}$ indicates that y = is to be differentiated twice.

Further differentiations give us the derivatives of higher than 2. These derivatives are defined and denoted a lows*:

* In practice, the symbol y'' (read y double prime) and y''' (read y triple prim used in place of $y^{(2)}$ and $y^{(3)}$ respectively. Similarly, f'' (x) and f''' (x) are used inste $f^{(2)}(x)$ and $f^{(3)}(x)$ respectively. The symbol y'''' should not be used in place of y($f^{(4)}(x)$.

$$\frac{d^3y}{dx^3} = f'''(x) = y''' = D_x^3 y \qquad \textbf{3rd derivative}$$

$$\frac{d^4y}{dx^4} = f^{(4)}(x) = y^{(4)} = D_x^4 y \qquad \textbf{4th derivative}$$

$$\frac{d^n y}{dx^n} = f^{(n)}(x) = y^{(n)} = D_x^n y \qquad \textbf{nth derivative}$$

...te that parenthesis are used in $y^{(n)}$ and $f^{(n)}(x)$. The ...l $y^{(n)}$ is used to distinguish it from the symbol y^n. Re- ...hat y^n indicates the *nth power* of $y = f(x)$ while the pre- ...otation $y^{(n)}$ indicates the *nth derivative* of $y = f(x)$. The ...eason holds for the symbol $f^{(n)}(x)$.

...PLE: If $y = x^4 - 2x^3 + 5x^2 - 4$, then

$$\frac{dy}{dx} = y' = 4x^3 - 6x^2 + 10x$$

$$\frac{d^2y}{dx^2} = y'' = 12x^2 - 12x + 10$$

$$\frac{d^3y}{dx^3} = y''' = 24x - 12$$

$$\frac{d^4y}{dx^4} = y^{(4)} = 24$$

$$\frac{d^5y}{dx^5} = y^{(5)} = 0$$

EXERCISE 2.4

...d the second and third derivative of each of the following:

1. $y = x^5 + 3x^{-2} + 4x$
2. $y = \frac{1}{x}$
3. $y = \sqrt{4 - x^2}$

4. $y = \dfrac{4x}{x+1}$

5. $y = (x+5)^2$

6. $y = \left(a^{\frac{1}{2}} - x^{\frac{1}{2}}\right)^2$

7. $y = \dfrac{1+\sqrt{x}}{\sqrt{x}}$

8. $y = \dfrac{x}{\sqrt{x-1}}$

9. $y = \dfrac{x^2}{x+1}$

10. If $y = \sqrt[3]{x}$, find $f'(8)$ and $f''(8)$.

11. If $y = x^5$, find y^4 and $y^{(4)}$.

12. Find the point on the curve $y = x^3 + 3x$ for w
$y' = y''$.

13. How fast does the slope of the curve $y = (x^2 + x +$
change at the point where $x = 2$?

14. Find the rate of change of the slope of the c
$y = x^3 - 1$ at $(2, 7)$.

2.8 Implicit Differentiation

In the preceding sections, we have been concerned m
with functions defined by the equation

$$y = f(x).$$

In this form, y is said to be an *explicit function** of x. For example, in the equation $y = x^2 + 4x + 3$, y is an explicit function of x.

If y is a function of x but is not expressed explicitly in terms of x, then y is said to be an *implicit function* of x. In each of the equations below, y is an implicit function of x.

1. $x^2 + 4xy + 4y^2 = 0$

2. $2 - (1 - x)\ln y = 0$

3. $y^2 = 4x^2 + 9$

4. $\sqrt{x+y} + xy = 21$

5. $e^x = \cos y$

Equations (3), (4) and (5) can be written in the form given by equations (1) and (2), i.e., the right member of the equation is zero. Then, in general, an implicit function may be represented by the equation

$$\text{E (2.3)} \qquad f(x, y) = 0$$

An implicit function given in the form E (2.3) can be converted to the form $y = f(x)$. For instance, the explicit form of the equation (3) above is $y = \sqrt{4x^2 + 9}$ while that of equation (5) is $y =$ Arccos e^x. The reader is urged to obtain the explicit forms of equations (1), (2) and (4). However, there are implicit functions which are quite difficult (and may be quite impossible) to convert to their corresponding explicit forms. Thus finding $\frac{dy}{dx}$ from an *implicit relationship* between x and y is of particular importance in those cases where it is difficult (if not impossible) to obtain an explicit solution for y in terms of x.

To find $\frac{dy}{dx}$ or y' of an implicit function, we differentiate both sides of the equation with respect to x and then solve for $\frac{dy}{dx}$ or y'. The process involved is called *implicit differentiation.*

---

*If from $y = f(x)$, we solve for x in terms of y, we get the form $x = g(y)$. In this latter form, x is said to be an explicit function of y.

EXAMPLE 1. Find $\frac{dy}{dx}$ if $y^2 = 4x^2 + 9$

Solution: $\frac{d}{dx}(y^2) = \frac{d}{dx}(4x^2 + 9)$

$$2y\frac{dy}{dx} = 8x + 0$$

$$\frac{dy}{dx} = \frac{4x}{y}$$

EXAMPLE 2. Find y' if $x^2 + 4xy + 4y^2 = 0$.

Solution:
$$\begin{aligned} 2x + 4xy' + 4y + 8yy' &= 0 \\ 4xy' + 8yy' &= -2x - 4y \\ (4x + 8y)y' &= -(2x+4y) \\ y' &= \frac{-(x+2y)}{2x+4y} \end{aligned}$$

EXAMPLE 3. Find y'' if $x^2 + y^2 = 4$.

Solution: Differentiating with respect to x, we have

$$2x + 2yy' = 0$$

$$y^1 = \frac{-x}{y}$$

Differentiating further with respect to x,

$$y'' = \frac{y(-1) - (-x)\,y'}{y^2}$$

$$= \frac{-y \quad x\left(\frac{-x}{y}\right)}{y^2} \qquad \text{since } y' = -\frac{x}{y}$$

$$= \frac{-y^2 - x^2}{y^3}$$

$$= \frac{-(x^2 + y^2)}{y^3}$$

$$= \frac{-4}{y^3} \qquad \text{since } x^2 + y^2 = 4$$

Note that y'' can also be obtained without solving for y' in terms of x and y. That is, starting with $2x + 2yy' = 0$ or

$$x + yy' = 0$$

we can differentiate implicity again to obtain

$$1 + yy'' + y'y' = 0$$
$$1 + yy'' + (y')^2 = 0$$

Solving for y'', we get

$$y'' = \frac{-1 - (y')^2}{y}$$

Substituting $y = \frac{-x}{y}$ in the equation above and simplifying, we get $y'' = -\frac{4}{y^3}$

EXERCISE 2.5

Find $\frac{dy}{dx}$ by implicit differentiation

1. $x^3 + y^3 - 6xy = 0$
2. $x^2 + xy^2 + y^2 = 1$
3. $\sqrt{x+y} + xy = 21$
4. $\sqrt{x} + \sqrt{y} = \sqrt{a}$
5. $b^2x^2 + a^2y^2 = a^2b^2$
6. $(x - y)^3 = (x+y)^2$
7. $y = 4(x^2 + y^2)$

8. $y^2 = \left(\frac{3x+1}{2x-3}\right)$

9. $y^2 - 3x + 2y = 0$

10.

Find y" in each of the following:

11. $xy = 32$

12. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

13. $y^2 - 16x = 0$

14. $x^2 - 2xy + 3y^2 = 4$

15. $4x^2 + 9y^2 = 36$

Find the slope of the curve at the given point.

16. $2x^3 + 2y^3 = 9xy$ at $(2, 1)$

17. $y^3 = x^2 - 1$ at $(3, 2)$

18. $x^2 + 4\sqrt{xy} + y^2 \quad 25$ at $(4, 1)$

19. $x^3 + x^2y + y^3 = 9$ at $(-1, 2)$

20. $\sqrt{3x} + \sqrt[3]{4y} = 5$ at $(3, 2)$

21. A circle is drawn with its center at (8, 0) and with radius r such that the circle cuts the ellipse $x^2 + 4y^2 = 16$ at right angles. Find the radius of the circle.

22. The vertex of the parabola $y^2 = 8x$ is the center of an ellipse. The focus of the parabola is an end of the minor axis of the ellipse, and the parabola and ellipse intersect at right angles. Find the equation of the ellipse.

hapter

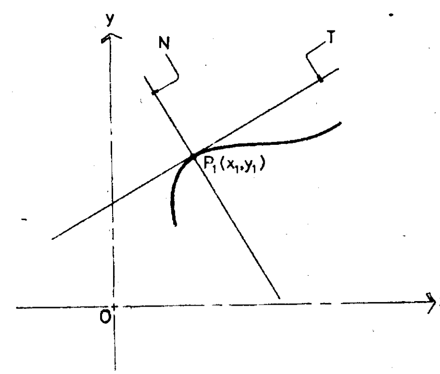
3

Some Applications of the Derivative

The *derivative* is a powerful tool in the solution of many ...ms in science, engineering, geometry and economics. Among ... problems which you will find not only useful but also quite ...ting are those situations which call for maximizing or mini... a function. For instance, a manufacturer is interested in ...ing his cost of production. An engineer may want to ...mine the dimensions of the strongest rectangular beam that ... cut from a circular log of known diameter. A farmer may ... to find the area of the largest rectangular field which he can ... with a given amount of fence. We shall find that the ...tive is a very useful aid in solving such types of problems. ... this chapter will introduce the students to some applica... of the derivative.

...uations of Tangents and Normals

In section 2.3, we have seen that the derivative of a function ... interpreted as the slope of the tangent to the graph of the



* A ...rprising application of the derivative is a technique called Newton's method. ... enables one to find the zeros of a function as accurately as desired. This is ... in this book.

function. In Fig. 3.1, the line T is the *tangent* to the curve y = at P_1 (x_1, y_1). The other line N perpendicular to T at P_1 is the *normal* to the curve.

If $y = f(x)$ is differentiable at x_1, i.e., $f'(x_1)$ exists, we may formulate the following definitions about the tangen normal to the curve $y = f(x)$.

DEFINITION 3.1 The tangent to the curve $y = f(x)$ (x_1, y_1) is the line through P_1 with $f'(x_1)$.

DEFINITION 3.2 The normal to the curve $y = f(x)$ (x_1, y_1) is the line through P_1 and p dicular to the tangent at P_1.

The equation of the tangent is given by the point-slope of the equation of a straight line in analytic geometry, that is

$$E(3.1) \qquad y - y_1 = m(x - x_1)$$

where m = value of y' at $x = x_1$ or $m = f'(x_1)$. Since the n is perpendicular to the tangent, then its slope is the negativ procal of the slope of the tangent. Hence the equation of th mal is

$$E(3.2) \qquad y - y_1 = -\frac{1}{m}(x - x_1)$$

where as defined above $m = f'(x_1)$.

EXAMPLE: Find the equations of the tangent and norm the curve $y = x^3$ at the point (2, 8).

Solution: The point of tangency is (2, 8). Hence $x_1 =$ $y_1 = 8$. Since $y' = 3x^2$, then $m = 3(2)^2 = 12$.

By E(3.1), the equation of the tangent is

$$y - 8 = 12(x - 2)$$

or $\qquad 12x - y - 16 = 0$

By E(3.2), the equation of the normal is

$$y - 8 = \frac{-1}{12}(x - 2)$$

$$x + 12y - 98 = 0$$

EXERCISE 3.1

The equations of the tangent and normal to the graph of the function at the given point.

1. $y = 3x^2 - 2x + 1$, (2, 9)
2. $y = 1 + 3\sqrt{x}$, (4, 7)
3. $y = x\sqrt{x-1}$, (5, 10)
4. $y^2 = \dfrac{x^3}{4-x}$, (2, 2)
5. $y = \dfrac{2}{x}$, (1, 2)

Where will the tangent to $y = \sqrt{4x}$ at (1, 2) cross the x-axis?

At what point on the curve $xy^2 = 6$ will the normal pass through the origin?

Find the area of the triangle formed by the coordinate axes and the tangent to $xy = 5$ at (1, 5).

Find the area of the triangle bounded by the coordinate axes and the tangent to $y = x^2$ at the point (2, 4).

Find the area of the triangle formed by the x-axis, the tangent and normal to $xy = 4$ at (2, 2).

Find the tangent to $x^2 + y^2 = 5$ and parallel to $2x - y = 4$.

Find a normal of slope $\frac{1}{3}$ to the curve $y^2 = 2x^3$ -

13. Show that the tangent with slope m to $y^2 = 4ax$ is the
$y = mx + \frac{a}{m}$

3.2 Angle Between Two Curves

The angle between two curves at a point of interse may be defined as the *angle between their tangents at this p of intersection.* If the tangents are not perpendicular to other, then such tangents form a pair of acute angles and a p obtuse angles. The acute and obtuse angles are supplementary

For a general discussion of this concept, consider the c $y = f_1(x)$ and $y = f_2(x)$ which intersect at a point P_o $(x_o$ as shown in Fig. 3.2. Let θ_1 and θ_2 be the inclinations o tangents T_1 and T_2 at P_o respectively. Let ϕ be the ang tween these tangents. Then, by definition, ϕ is also the between the curves. It can easily be shown that ϕ, θ_1, an are related by the equation

$$\phi = \theta_2 - \theta_1 \qquad (1)$$

Then taking the tangent of both sides of (1), we get

$$\tan\phi = \tan(\theta_2 - \theta_1) \qquad (2)$$

or

$$\tan\phi = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_2} \qquad (3)$$

Let m_1 and m_2 be the slopes of T_1 and T_2 respectively. $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$. Substituting these in equ (3) above, we obtain

$$\tan\phi = \frac{m_2 - m_1}{1 + m_2 m_1} \qquad (4)$$

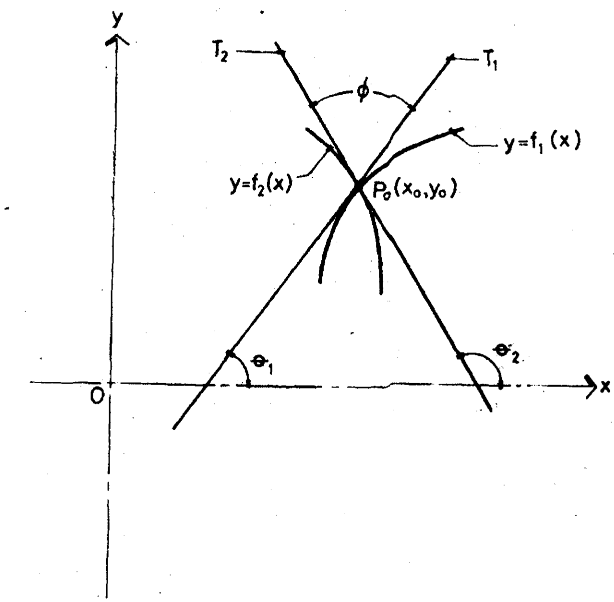


FIG. 3.2

The sign of $\tan\phi$ in (4) is positive or negative depending upon the values of m_1 and m_2 or on the order in which m_1 and m_2 are used. If $\tan\phi > 0$, then ϕ is acute and if $\tan\phi < 0$, then ϕ is obtuse. In most books, it is customary to find only the acute angle of interection between the curves. The same is true in this book. Since $\tan\phi > 0$ if ϕ is acute, then we may use the absolute value symbol in the right member of (4). Thus our final formula would be

$$\text{E (3.3)} \qquad \tan\phi = \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right|$$

where the values of m_1 and m_2 are given by the derivatives of the functions at P_0 (x_0, y_o). That is,

$$m_1 = \frac{d}{dx}\left[f_1(x)\right] \text{ at } P_o$$

$$m_2 = \frac{d}{dx}\left[f_2(x)\right] \text{ at } P_o$$

EXAMPLE: Find the acute angle of intersection between the curves $x^2 = 8y$ and $xy = 8$.

Solution: Solving the given equations simultaneously, we get $x = 4$ and $y = 2$. Hence the point of intersection is $P_0\ (x_0, y_0) = (4, 2)$ as shown in Fig. 3.3.

Differentiating the first equation $x^2 = 8y$, we get

$$\frac{dy}{dx} = \frac{x}{4} = m_1$$

Similarly, differentiating the second equation $xy = 8$, we get

$$\frac{dy}{dx} = \frac{-8}{x^2} = \frac{-xy}{x^2} = \frac{-y}{x} = m_2$$

Therefore at the point (4, 2), we have

$$m_1 = \frac{4}{4} = 1$$

$$m_2 = -\frac{2}{4} = -\frac{1}{2}$$

Then by E (3.3), we obtain

$$\tan\phi = \left|\frac{-\frac{1}{2} - 1}{1 + (-\frac{1}{2})(1)}\right|$$

$$= |-3| = 3$$

Hence, $\phi = \text{Arctan } 3$

$$\phi = 71°34'$$

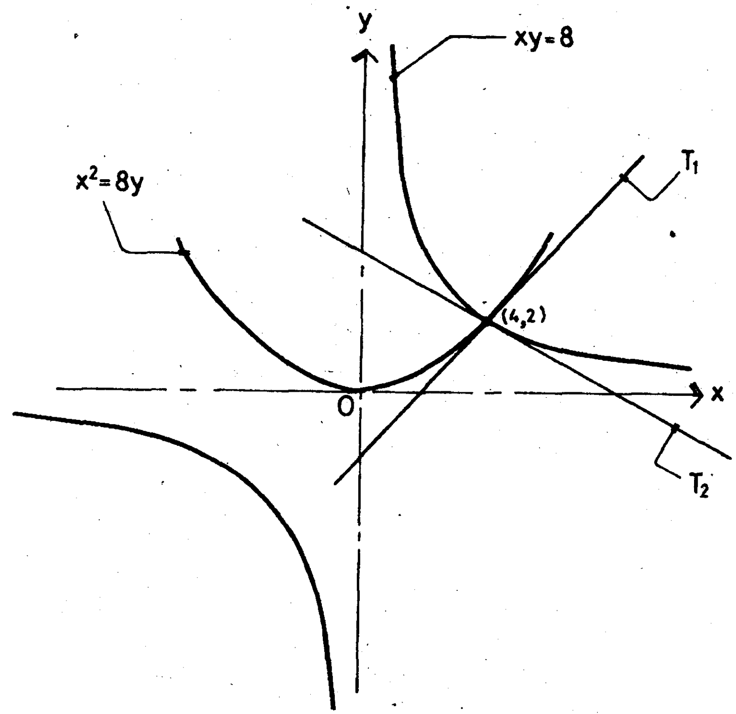


FIG. 3.3

EXERCISE 3.2

Find the acute angle between the given curves.

1. $y^2 = 2x$ and $4x^2 + 4y^2 + 5y = 0$
2. $x^2 + y^2 = 5$ and $y^2 = 4x + 8$
3. $x^2y + 4a^2y = 8a^3$ and $x^2 = 4ay$
4. $2y^2 = 9x$ and $3x^2 = -4y$
5. $x^2y + 4y = 8$ and $x^2y = 4$
6. $xy = 18$ and $y^2 = 12x$

$I_2 = [1, 3]$

$I_3 = [3, \infty)$

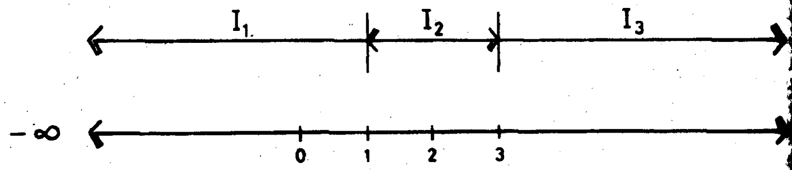


FIG. 3.5

In each of these subintervals, f(x) is either i
sing or decreasing. In the subinterval I_1, if
(choose any convenient value of x within t
binterval), we see that $f'(x) > 0$ and
quently f(x) is increasing in I_1. If $1 < x$
$f'(x) < 0$ and f(x) is decreasing in I_2. If
$f'(x) > 0$ and f(x) is increasing in I_3. The
of the function is shown in Fig. 3.6

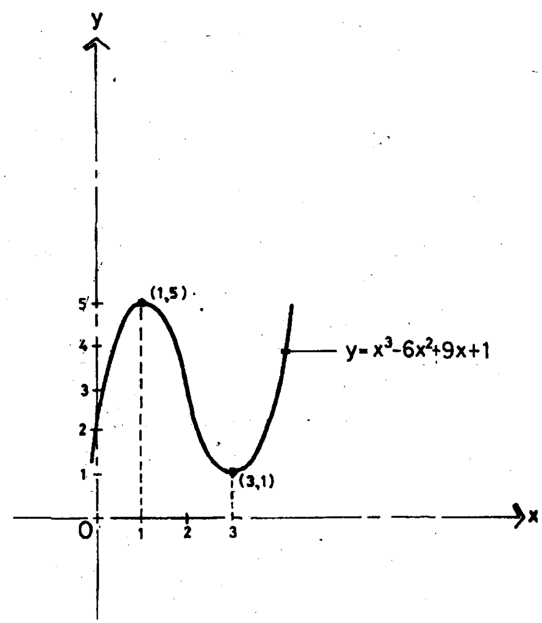


FIG. 3.6

EXERCISE 3.3

the interval or intervals where the function is increasing and
it is decreasing.

1. $f(x) = 2x^3 + 3x^2 - 36x$
2. $f(x) = x^3 - 3x + 3$
3. $f(x) = (x^2 - 9)^2$
4. $f(x) = 3x^2 - 6x - 9$
5. $f(x) = x^3 - 6x^2 + 4$
6. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 12$

...ine whether the function is increasing or decreasing in the
interval.

7. $f(x) = \sqrt{x}, \quad (1, 4)$
8. $f(x) = x^2 - 4, \quad [-2, 3]$
9. $f(x) = 6x - x^2, \quad (-1, 3]$
10. $f(x) = x^8 - 4x^3 + 2x, \quad [0, 2]$
11. $f(x) = 6x + 3x^2 - 4x^3, \quad [1, 3]$
12. $f(x) = 3x - 3x^2, \quad [\frac{1}{2}, 2]$

...imum and Minimum Values of a Function

Suppose $y = f(x)$ is a function which is continuous for all
...its domain. Let the graph of this function be represented by
...graph shown in Fig. 3.4 in the preceding section. For the
arc ... of the curve, we observe that

$$\text{between A and B}, \qquad f'(x) > 0$$

Next, we consider x = 3. Following the procedure above, we find that

when $x < 3$, $y' = (+)(-) = (-)$ or $y' < 0$

when $x > 3$, $y' = (+)(+) = (+)$ or $y' > 0$

This satisfies (2) of FDT and therefore, y is a minimum at x = 3.

(b) Substituting x = 1 in $y = x^3 - 6x^2 + 9x - 3$, we get y = 1 which is the maximum value of the function. Likewise, substituting x = 3, we get y = − 3. This is the minimum value of the function.

(c) Therefore, the maximum point is (1, 1) and the minimum point is (3, − 3). The graph of the function is shown in Fig. 3.7.

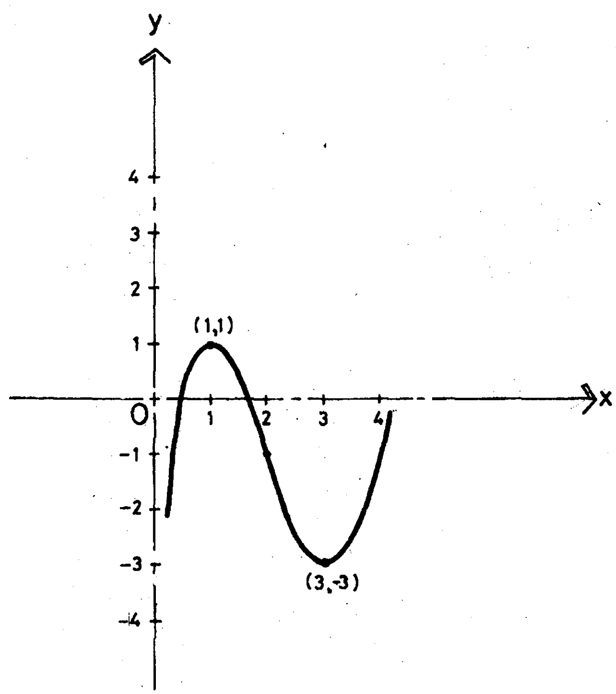


FIG. 3 7

EXERCISE 3.4

Find the value or values of x for which the given function has a maximum or a minimum value

1. $y = 8x^3 - 9x^2 + 1$
2. $y = x^3 - 4x^2 + 4x$
3. $y = 4x^{-1} + x$
4. $y = x^4$
5. $y = x^3 - 3x^2 + 3x$
6. $y = \dfrac{x^2 + 1}{x}$
7. $y = x^2 (x - 1)^2$
8. $4y = 3x^4 - 16x^3 + 24x^2$
9. $3y = x^3 + 3x^2 - 9x + 3$
10. $y = \dfrac{x^2 - 4x + 5}{x - 2}$
11. $y = x^3 - 3x^2 + 3$
12. $y = x^3 - 6x^2 + 9x + 3$
13. $y = 2x^3 - 9x^2 + 12x + 4$
14. $y = (x - 2)^4$

3.5 Significance of the Second Derivative

In section 3.4, it was shown how the sign of the first derivative or y′ of a function y = f(x) determines whether the function is a maximum or a minimum at a critical value of x in a given interval. Now we shall show how the sign of the second derivative or y'' may be used for the same purpose.

Consider again the graph in Fig. 3.4 but this time with points Q and R added as shown in Fig. 3.8. For the part ABQ, we observe that the curve always lies below its tangent. It is customary to say, in this case, that the curve is *concave downward* (as seen from below). It is clearly seen, likewise, that for the part QCR, the curve always lies above its tangent. Here, we say that the curve is *concave upward* (as seen from above).

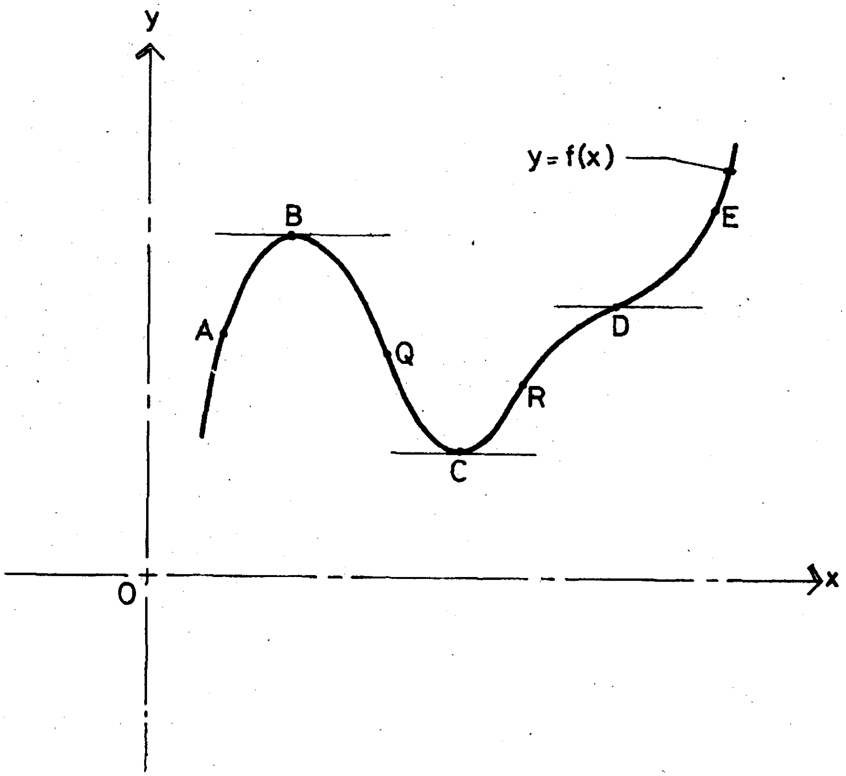


FIG. 3.8

Now, we recall that by definition

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$\frac{dy}{dx}$ is the slope of the curve or the tangent at any point, then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\text{(slope)}$$

$\frac{d^2y}{dx^2}$ or y'' measures the rate of change of the slope of a curve. that along ABQ of the curve in Fig. 3.8, $\frac{d}{dx}$ (slope) is negative, while along QCR, $\frac{d}{dx}$ (slope) is positive. Hence, we conclude

(i) The graph of $y = f(x)$ is concave upward if $y'' > 0$ and concave downward if $y'' < 0$.

If B is a maximum point and C is a minimum point, then we conclude that

(ii) The graph of $y = f(x)$ is concave downward at a maximum point and concave upward at a minimum point.

From (i) and (ii), we may formulate a test for determining whether a function $y = f(x)$ is a maximum or a minimum at a critical value of x.

Second Derivative Test (SDT)

(i) The function $y = f(x)$ is a maximum at $x = a$ if $f'(a) = 0$ and $f''(a) < 0$.

(ii) The function $y = f(x)$ is a minimum at $x = a$ if $f'(a) = 0$ and $f''(a) > 0$.

Note that if $f''(a) = 0$ or if $f''(a)$ does not exist, then SDT fails; under this particular situation, we may use FDT.

EXAMPLE 1: Find the value of x for which the function $y = x^3 - 6x^2 + 9x - 3$ is a maximum or a minimum.

Solution: Since $y = x^3 - 6x^2 + 9x - 3$

then $y' = 3x^2 - 12x + 9 = 3(x-1)(x$

and $y'' = 6x - 12$

Setting $y' = 0$, we get $x = 1$ and $x = 3$. note that

when $x = 1$, $y'' < 0$

when $x = 3$, $y'' > 0$

Therefore, the function is a maximum and a minimum at $x = 3$. The results h with the results in the example given i 3.4

A point where the sense of concavity changes is *point of inflection.* For instance, in Fig. 3.8, consider The curve to the left of Q is *concave downward* wh right, the curve is *concave upward.* Hence Q is a point tion. Like the maximum and minimum points, the poi flection is an essential feature of a curve when one is tr graph of a function.

It can be shown that if $y = f(x)$ has a point of infl $x = a$, then $f''(a) = 0$ or $f''(a)$ does not exist*. To det the curve of the function has a point of inflection at value, we may use any of the following tests:

POINT OF INFLECTION TESTS (PIT)

(1) If $f''(a) = 0$ and if $f'(x) \gtrless 0$ for $x < a$ and f for $x > a$, then $y = f(x)$ has a point of infl $x = a$.

(2) If $f''(a) = 0$ and if $f'''(a) \neq 0$, then $y = f(x)$ ha of inflection at $x = a$.

*This fact is stated as a theorem in some books on calculus.

words, statement (1) says that *$y = f(x)$ has a point of in-* *at a if the second derivative is zero at $x = a$ and the derivative changes sign as the value of x increases through* ...nt (2) states that *if the second derivative is zero but the* ...*ative is not equal to zero at $x = a$. then $y = f(x)$ has a inflection at $x = a$*

...2: Find the values of x for which the curve of $y = x^4 - 4x^3$ has points of inflection.

...tion: $y' = 4x^3 - 12x^2 = 4x^2(x - 3)$

$y'' = 12x^2 - 24x = 12x(x - 2)$

Setting $y'' = 0$, we get $x = 0$ and $x = 2$. Now we test these values.

(a) *By statement (1)*

For $x = 0$: when $x < 0$, $y'' > 0$

when $x > 0$, $y'' < 0$

For $x = 2$: when $x < 2$, $y'' < 0$

when $x > 2$, $y'' > 0$

Since the sign of y'' changes in either case, then the curve of the function has points of inflection at $x = 0$ and $x = 2$.

(b) *By Statement (2)*

Differentiating further y'', we get

$y''' = 24x - 24 = 24(x - 1)$.

When $x = 0$, $y''' = -24 \neq 0$

When $x = 2$, $y''' = 24 \neq 0$

Since $y''' \neq 0$, then the curve of the function has points of inflection at such values of x.

EXERCISE 3.5

Find the value (or values) of x for which the curve of the function has a point of inflection.

1. $y = (x-1)^4 (x-6)$
2. $y = 2x^3 - 3x^2 - 36x + 25$
3. $y = 3x^4 - 4x^3 + 1$
4. $y = x^4 - 4x^3 + 4x^2$
5. $y = 3x^5 - 15x^4 + 20x^3 + 3$

Find the maximum, minimum or inflection point of each given curves. Sketch the graph.

6. $y = x^3 - 3x^2 + 4$
7. $4y = 3x^4 - 16x^3 + 24x^2$
8. $3y = x^3 + 3x^2 - 9x + 3$
9. $y = \dfrac{2}{x\ 1}$

11. $y = \dfrac{6x}{x^2 + 3}$

11. $y = 5x - x^5$

12. $y = \dfrac{1}{4}x^3 - \dfrac{5}{2}x^2$

13. $y = \dfrac{4x}{x^2 + 4}$

3.6 Applications of Maxima and Minima

The methods of determining the maximum or mi value of a function in the preceding section find many

tions in a surprisingly wide variety of problems in science, engineering, geometry, economics, and other disciplines concerned with maxima and minima. These problems, whether they are of practical importance or simply of theoretical interest, are often referred to as "*max-min*" problems. In solving problems of this type, no general rule applicable in all cases can be given. However, the reader may find the following steps possibly helpful:

1. Draw a figure whenever necessary and denote the variable quantities by x, y, z, etc.
2. Identify the quantity to be maximized or minimized and express it in terms of other variable quantities. If possible, express this quantity in terms of one independent variable.
3. Find the first derivative of the function and set it to zero. (why?) The roots of the resulting equation are the critical numbers which will give the desired maximum or minimum value of the function. (Note: The critical number which gives a maximum or a minimum value may be verified by SDT. However, in practice, the desired value can be selected at once by *inspection.*)

EXAMPLE 1: A long strip of tin 30 cm wide is to be made into a gutter with rectangular cross section by turning up equal widths along the edges. Find the depth of the gutter which yields the greatest carrying capacity.

Solution: Let x = depth of the gutter (Fig. 3.9)

y = base of the rectangular cross section

A = area of the rectangular cross section

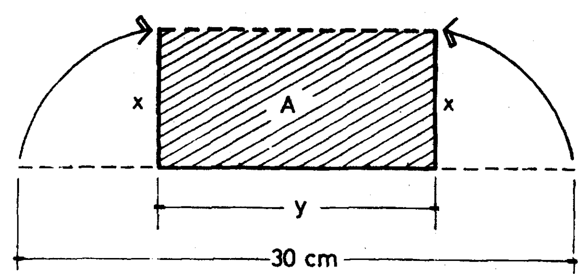


FIG. 3.9

To insure the greatest carrying capacity, we must make the area of the cross section as great as possible. That is, we maximize A. Thus

$$A = xy \quad (1)$$

But $\quad 2x + y = 30 \qquad (2)$

From (2), we get

$$y = 30 - 2x \qquad (3)$$

Substituting (3) in (1), we obtain

$$A = 30x - 2x^2 \qquad (4)$$

Differentiating (4) with respect to x,

$$\frac{dA}{dx} = 30 - 4x$$

Setting $A = 0$,

$$30 - 4x = 0$$

$$x = 7.5 \text{ cm}$$

EXAMPLE 2: A closed cylindrical tank (Fig. 3.10) is to be made with a fixed volume, Find the relative dimensions of the tank which will require the least amount of material in making it.

Solution: This problem amounts to finding the relation between the height h and the radius r of the tank of minimum surface area and fixed volume.

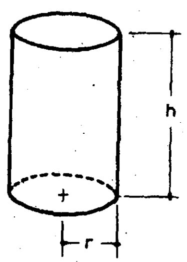


FIG. 8.10

Let A = total surface area of the tank
A_t = area of the top = πr^2
A_b = area of the bottom = πr^2
A_s = area of the side = $2\pi rh$
V = volume of the tank (constant)

The quantity to be minimized is A. Thus

$$A = A_t + A_b + A_s$$

or $\quad A = 2\pi r^2 + 2\pi rh \qquad (1)$

The volume of the cylindrical tank is

$$V = \pi r^2 h \qquad (2)$$

Differentiating (1) with respect to r,

$$\frac{dA}{dr} = 4\pi r + 2\pi r\frac{dh}{dr} + 2\pi h$$

Setting $\frac{dA}{dr} = 0$ and solving for $\frac{dh}{dr}$, we get

$$\frac{dh}{dr} = \frac{2r - h}{r} \qquad (3)$$

Similarly, differentiating (2) with respect to r, we obtain

$$0 = \pi r^2 \frac{dh}{dr} + 2\pi rh \qquad \text{(note: V is constant)}$$

and solving for $\frac{dh}{dr}$, we get

$$\frac{dh}{dr} = -\frac{2h}{r} \qquad (4)$$

Equating (3) and (4), we have

$$\frac{-2r - h}{r} = \frac{-2h}{r}$$

from which we obtain the relation h = 2r. The result tells us that the proportion which requires the least amount of material in making the tank with a fixed volume is that the *height should be twice the radius of the base.*

ALTERNATIVE SOLUTION: Another solution is to reduce the function to be made a minimum as a function of a single variable. Thus in this problem, we may express A in terms of the variable r. We start with the two equations (1) and (2) given above. That is,

$$A = 2\pi r^2 + 2\pi rh \qquad (1)$$

$$V = \pi r^2 h \qquad (2)$$

From (2), solve for h.

$$h = \frac{V}{\pi r^2} \qquad (3)$$

Substitute (3) in (1) and simplify to

$$A = 2\pi r^2 + \frac{2V}{r} \qquad (4)$$

Differentiating (4) with respect to r, keeping in mind that V is a constant, we get

$$\frac{dA}{dr} = 4\pi r - \frac{2V}{r^2}$$

Setting $\frac{dA}{dr} = 0$,

$$4\pi r - \frac{2V}{r^2} = 0$$

$$\text{or} \qquad 2\pi r^3 = V \qquad (5)$$

Substitute (2) in (5). We get

$$2\pi r^3 = \pi r^2 h$$

$$h = 2r$$

which agrees with the result of our first solution above.

EXERCISE 3.6

A closed right circular cylindrical tank is to have a capacity of 128π m^3. Find the dimensions of the tank that will require the least amount of material in making it.

volume of an open box with a square base is 4,000 cm^3. the dimensions of the box if the material used to make box is a minimum.

the volume of the largest right circular cylinder that can cut from a circular cone of radius 6 cm and height 9 cm.

the height of the right circular cylinder of maximum volume that can be inscribed in a sphere of radius 15 cm.

the dimensions of the largest rectangle that can be inscribed in the ellipse $9x^2 + 16y^2 = 144$. The sides of the rectangle are parallel to the axes of the ellipse.

isosceles trapezoid has a lower base of 16 cm and the sides are each 8 cm. Find the width of the upper for greatest area.

trapezoidal gutter is to be made from a sheet of tin 22 cm by bending up the edges. If the base is 14 cm wide, width across the top gives the greatest carrying capacity?

sum of the bases and altitude of an isosceles trapezoid is cm. Find the altitude if the area is to be a maximum.

building with a rectangular base is to be constructed on a in the form of a right triangle with legs 18 m and 24 m. the building has one side along the hypotenuse of the triangle find the dimensions of the base of the building for maximum floor area.

rectangular field is to be enclosed and divided into four lots by fences parallel to one of the sides. A total 10,000 meters of fence are available. Find the area of the field that can be enclosed.

room of floor area 18 m^2 is divided into six cubicles of floor area by erecting two wooden partitions 2 m high parallel to one wall and another partition 2 m high parallel another wall. Find the dimensions of the room if the amount of wood is used.

Norman window, consisting of a semicircle surmounting a rectangle, has a given perimeter. Find the radius of the semicircle to admit the most light.

right circular cone of radius R and altitude H is circumscribed about a sphere of radius r. Find the relation between

H and r if the volume of the cone is to be a minimum.

14. Find the lengths of the sides of an isosceles triangle given perimeter if its area is to be as great as possible.

15. An oil can with a given volume is made in the shap cylinder surmounted by a cone. If the radius r of the equal to 3/4 of its altitude h find the relation betwee the height H of the cylinder for minimum surface area can.

16. The sector of a circle of radius r has a given perim Show that $L = 4r$ for maximum area of the

17. A rectangle is inscribed in the ellipse $b^2x^2 + a^2y^2$ with each of its sides parallel to an axis of the ellip the greatest perimeter which the rectangle can have.

18. A wire of length L is cut into two pieces, one of bent into the shape of a circle and the other into th of an equilateral triangle. Find the length of each that the sum of the enclosed areas is a minimum.

19. Find the length of the longest beam that can be mo rizontally from a corridor of width a into a corr width b if the two corridors are perpendicular to eac

20. A man in a boat 6 km from the nearest point P on a shore wishes to reach a point Q down the straight sh 12 km from P. On water, he can travel 4 km/hr and 5 km/hr. How far from P should he land in order mize his total travel time?

21. A line is drawn perpendicular to the x-axis cutting rabola $y = 4x - x^2$ and the line $y = 12 - 2x$ at P and L respectively. Find the value of x which m distance from L to P a minimum.

22. The upper and lower vertices of a rectangle lie on th $x^2 = 5 - y$ and $x^2 = 4y$ respectively. The sides of tangle are parallel to the coordinate axes. Find the m area of the rectangle.

23. The points (3, 2) and (1, 6) lie on the ellipse $y^2 + 4x$ Find a point on the ellipse so that the area of the having these three points as vertices is a maximum.

24. Find the point on the curve $y = x^3$ which is nea point (4, 0).

Find the point on the curve $y = \frac{2}{3}\sqrt{18 - x^2}$ (first-quadrant) where a tangent may be drawn so that the area of the triangle formed by the tangent line and the coordinate axes is a minimum.

A rectangular field is to be fenced, one side of which is the bank of a straight river. It is given that the material for the two opposite sides costs ₱3.00 per meter and the material for the side opposite the river costs ₱6.00 per meter. If an amount of ₱600.00 is available, what should the dimensions be to enclose a maximum area?

A manufacturer of a certain brand of appliance estimates that he can sell 5,000 units a year at ₱900.00 each and that he can sell 1,500 units more per year for each ₱100.00 decrease in price. What price per unit will give the greatest returns?

A closed rectangular box whose base is twice as long as it is wide has a volume of 36,000 cm^3. The material for the top costs 10 centavos per sq. cm.; that for the sides and bottom costs 5 centavos per sq. cm. Find the dimensions that will make the cost of making the box a minimum.

A mango-grower observes that if 25 mango trees are planted per hectare, the yield is 450 mangoes per tree and that the yield per tree decreases by 10 for each additional tree per hectare. How many trees should be planted per hectare to obtain the maximum crop?

Buses are to be charted for an excursion. The bus company charges ₱20.00 per ticket if not more than 200 passengers go with the trip. However, the company agrees to reduce the price of every ticket ₱0.05 for each passenger in excess of 200 passengers. What number of passengers will produce the maximum gross income?

Related Rates

We recall that if $y = f(x)$, then $\frac{dy}{dx}$ is the rate of change of y with respect to x. Hence if $y = f(t)$, then $\frac{dy}{dt}$ is the rate of change of y with respect to t. If t denotes the time, then $\frac{dy}{dt}$ is simply referred to as the *time rate of change* of y. Likewise, $\frac{dx}{dt}$ is the time rate of change of x. These rates of change are related by the

E (3.4) $\qquad \dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$

For example, if $y = x^2 + 4x + 3$, then $\frac{dy}{dx} = 2x + 4$ and by $\frac{dy}{dt} = (2x + 4)\frac{dx}{dt}$. This equation is said to be the result of di tiating both sides of $y = x^2 + 4x + 3$ with respect to th Thus in practice, to find $\frac{dy}{dt}$ of the equation $y = f(x)$, we get the derivative of y with respect to x and then mult result by $\frac{dx}{dt}$.

Many physical problems deal with rates of change of ties with respect to time. For instance, when water is pou a tank, the water surface is rising with respect to time. Th change in the water level may be expressed in terms of th change of the depth of the water. If we denote this dep then $\frac{dh}{dt}$ is the time rate of change of the depth. Simil represents the volume, then $\frac{dV}{dt}$ is the time rate of chang volume. If $V = f(h)$, then by E (3.4), we have $\frac{dV}{dt} = \frac{dV}{dh}$ ·

In solving "*time rate*" *problems*, it is important to that all quantities which change with respect to time denoted by letters. *Do not substitute the numerical value variable until after differentiation with respect to the done**.

EXAMPLE 1: Water is poured into a conical tank 6 m a top and 8 m deep at the rate of 10 m^3/m fast is the water level rising when the the tank is 5 m deep?

Solution: (See Fig. 3.11) At time t, let

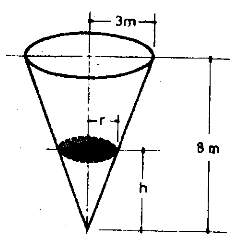


FIG. 3.11

r = radius of the water surface

h = depth of the water

V = volume of the water

It is given that $\frac{dV}{dt} = 10$ m^3/min required to find $\frac{dh}{dt}$ at the instant w 5 m.

*This error is commonly committed by the student.

The volume of the water in the tank at time t is

$$V = \frac{1}{3}\pi r^2 h \qquad (1)$$

Since we are to find $\frac{dh}{dt}$, then we have to express V as a function of h. In Fig. 3.11 and by similar triangles, we have

$$\frac{r}{3} = \frac{h}{8} \qquad (2)$$

Solving for r in (2), we get

$$r = \frac{3h}{8} \qquad (3)$$

Substituting (3) in (1) and simplifying, we obtain

$$V = \frac{3\pi h^3}{64} \qquad (4)$$

Differentiating (4) with respect to t

$$\frac{dV}{dt} = \frac{9\pi h^2}{64}\,\frac{dh}{dt} \qquad (5)$$

Substituting $\frac{dV}{dt} = 10$ and $h = 5$ in (5),

$$10 = \frac{225\pi}{64} \cdot \frac{dh}{dt} \qquad (6)$$

Solving for $\frac{dh}{dt}$ in (6), we obtain

$$\frac{dh}{dt} = \frac{128}{45\pi} \text{ m/min}$$

EXAMPLE 2: A ship *A* is 20 km west of another ship *B*. If *A* sails east at 10 km/hr and at the same time *B* sails north at 30 km/hr, find the rate of change of the distance between them at the end of $\frac{1}{2}$ hr.

Solution: (See Fig. 3.12) At time t, let

s = distance between the ships

x = distance traveled by ship A

y = distance traveled by ship B

where $x = 10t$ and $y = 30t$. Hence $\frac{dx}{dt} = 10$ and $\frac{dy}{dt} = 30$.

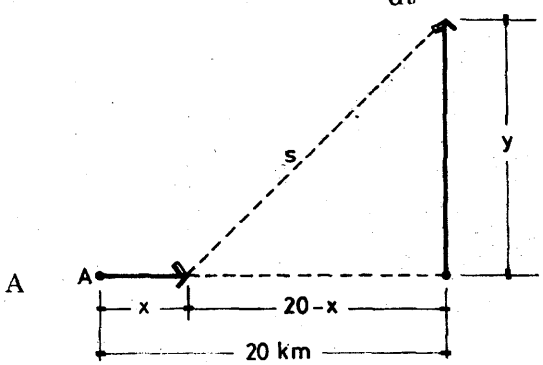


FIG. 3.12

It is required that we find $\frac{ds}{dt}$ when $t = \frac{1}{2}$ hr. Using the right triangle in Fig. 3.12, we get the relation

$$s^2 = (20 - x)^2 + y^2 \qquad (1)$$

Differentiating (1) with respect to plifying,

$$\frac{ds}{dt} = \frac{-(20-x)\,\frac{dx}{dt} + y\,\frac{dy}{dt}}{s} \qquad (2)$$

When $t = \frac{1}{2}$, we get

$$x = 10\left(\tfrac{1}{2}\right) = 5$$
$$y = 30\left(\tfrac{1}{2}\right) = 15$$

Solving for s in (1) and substituting these values of x and y, we have

$$s = \sqrt{(20 - x)^2 + y^2}$$

$$= \sqrt{(20-5)^2 + (15)^2}$$
$$= 15\sqrt{2}$$

Substituting the values of x, y, s, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in (2), we get

$$\frac{ds}{dt} = \frac{-(20-5)(10) + (15)(30)}{15\sqrt{2}}$$
$$= 10\sqrt{2}\ \text{km/hr}$$

Alternative Solution: Another approach is to express s in terms of t only. To obtain this, we substitute $x = 10t$ and $y = 30t$ in (1). Thus

$$s^2 = (20 - 10t)^2 + (30t)^2$$

or $$s = \sqrt{(20 - 10t)^2 + (30t)^2}$$

Differentiating

$$\frac{ds}{dt} = \frac{2(20-10t)(-10) + 2(30t)(30)}{2\sqrt{(20-10t)^2 + (30t)^2}}$$

Substituting $t = 1/2$, we get

$$\frac{ds}{dt} = 10\sqrt{2}\ \text{km/hr}$$

EXERCISE 3.7

1. The radius of a right circular cone is increasing at the rate of 6 cm/sec while its altitude is decreasing at 3 cm/sec. Find the rate of change of its volume when its radius is 8 cm and its altitude is 20 cm.
2. A ladder 6 m long leans against a vertical wall. The lower end of the ladder is moved away from the wall at the rate of 2 m/min. Find the rate of change of the area formed by the wall, the floor and the ladder when the lower end is 4 m from the wall.

3. A boy 5 ft tall is walking away from a street light at the rate of 3 ft/sec. If the light is 12 ft above the level ground, determine (a) the rate at which his shadow is lengthening, (b) the rate at which the tip of his shadow is moving and (c) the rate at which his head is receding from the light when he is 24 ft from the point directly below the light.
4. Water is running out of a conical tank 3 m across the top and 4 m deep at the rate of 2 m^3/min. Find the rate at which the level of water drops when it is 1 m from the top.
5. A reservoir is in the form of a frustrum of a cone with upper base of radius 9 ft and lower base of radius 4 ft and altitude of 10 ft. The water in the reservoir is x ft deep. If the level of the water is increasing at 4 ft/min, how fast is the volume of the water in the reservoir increasing when its depth is 2 ft? NOTE: The volume of a frustrum of a cone of upper base radius R and lower base radius r and height h is $V = \frac{1}{3}\pi h(R^2 + r^2 + Rr)$.
6. At noon, ship A is sailing due east at the rate of 20 km/hr. At the same time, another ship B, 100 km east of ship A, is sailing on a course 60° north of west at the rate of 10 km/hr. How fast is the distance between them changing at the end of one hr? When will the distance between them be least?
7. A ship is sailing north at 22 km/hr. A second ship sailing east at 16 km/hr crosses the path of the first ship 85 km ahead of it. How fast is the distance between them changing one hour later? When are they closest together?
8. Two roads intersect at 60°. A car 10 miles from the junction moves towards it at 30 mi/hr while a bus 10 miles from the junction moves away from it at 60 mi/hr. Calculate the rate at which the distance between the vehicles is changing at the end of 20 min.
9. A bridge is 10 m above a railroad track and at right angles to it. A train running at the rate of 20 m/s passes under the center of the bridge at the same instant that a car running 15 m/s reaches that point. How rapidly are they separating 3 seconds later?
10. A light at eye level stands 7 meters from a house and 5 meters from a path leading from the house to the street. A man walks along the path at 2 meters per second. Find the rate

at which his shadow moves along the wall of the house when he is 3 meters from the house.

A lamp post 3 m high is 6 m from a wall. A man 2 m tall is walking directly from the post toward the wall at 2.5 m/s. How fast is his shadow moving up the wall when he is 1.5 m from the wall?

The volume of a cube is increasing at the rate of 6 cm^3/min. How fast is the surface area increasing when the length of an edge is 12 cm?

Sand is poured at the rate of 10 m^3/min so as to form a conical pile whose altitude is always equal to the radius of its base. Find the rate at which the area of its base is increasing when the radius is 5 m.

A trough whose cross section is an equilateral triangle is 6 m long and 2 m wide across the top. If water is entering the trough at 15 m^3/min, at what rate is the water level rising in the trough when it is three-fourths full?

Water is poured into an inverted conical cistern of altitude 12 ft and radius of base 5 ft. If the water level rises at 36 ft/min, find the rate at which the dry surface of the inside of the cone is decreasing when the water is 4 ft deep.

A spherical iron ball 8 cm in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the uniform rate of 10 cm^3/min, how fast is the thickness decreasing at the instant when it is 2 cm thick?

Water flows out of a hemispherical tank at a rate which is 4 times the square root of its depth. If the radius of the tank is 8 ft, how fast is the water level falling when the water is 4 ft deep? Hint: Use the formula for the volume of a spherical segment, i. e. $V = \frac{1}{3}\pi h^2(3r - h)$

Rectilinear Motion

A body which moves in a straight line is said to be moving in *rectilinear motion*. If the moving body is small in comparison with the distance it covers, then it is customarily referred to as a particle.

Let s be the directed distance of a particle P from a fixed point O on a coordinate line (Fig. 3.13). If the motion of P along

the line is given by the equation $s = f(t)$, then the velocit
the acceleration a are defined as follows:

$$E\ (3.5) \qquad v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

$$E\ (3.6) \qquad a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

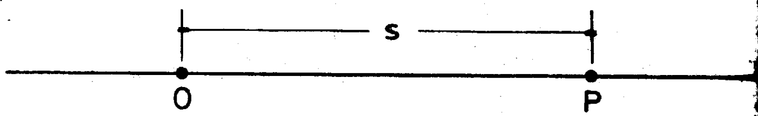


FIG. 3.13

Thus, we note that the *velocity is the time rate of ch* *the distance* while the *acceleration is the time rate of ch* *the velocity*. The absolute value of the velocity is called th of the particle.

The *sign of the velocity* determines the direction of of a particle P relative to its starting point. It can be sho

(a) If $v > O$, the particle P is moving to the right.

(b) If $v < O$, the particle P is moving to the left.

The *sign of the acceleration* determines whether the v increases with the time. It can also be shown that

(a) If $a > O$, the velocity v is increasing.

(b) If $a < O$, the velocity v is decreasing.

EXAMPLE 1: The motion of a particle moving on a coo line is given by

$$s = t^3 - 6t^2 + 9t + 3$$

Describe and diagram the rectilinear mot $t \geq O$.

Solution: $v = \dfrac{ds}{dt} = 3t^2 - 12t + 9 = 3\,(t-1)\,(t-3)$

$a = \dfrac{dv}{dt} = 6t - 12 = 6\,(t-2)$

Hence $v = O$ when $t = 1$ and $t = 3$

and $a = O$ when $t = 2$

We observe that

(a) When $t < 1$, $v > 0$ and $a < 0$. Hence during the time $t < 1$, the particle is moving to the right with decreasing velocity.

(b) When $1 < t < 2$, $v < 0$ and $a < 0$. Hence during this time interval, the particle is moving to the left with decreasing velocity.

(c) When $2 < t < 3$, $v < 0$ and $a > 0$. Hence during this time interval, the particle is moving to the left with increasing velocity.

(d) Finally, when $t > 3$, $v > 0$ and $a > 0$. Hence during the time interval $t > 3$, the particle is moving to the right with increasing velocity.

The motion decribed above is shown schematically in Fig. 3.14. Note that when $t = 0$, the particle is at $s = 3$ and moving to the right with $v = 9$. The particle continues to move to the right until $t = 1$ when $v = 0$. Since $s = 7$ when $t = 1$, then the particle momentarily stops after moving 4 units to the right of its original position when $t = 0$. It then reverses direction and moves to the left until $t = 3$. When $t = 3$, $v = 0$ and $s = 3$.

Thus it comes to stop again upon reaching its original position. Then it turns right and moves off to infinity.

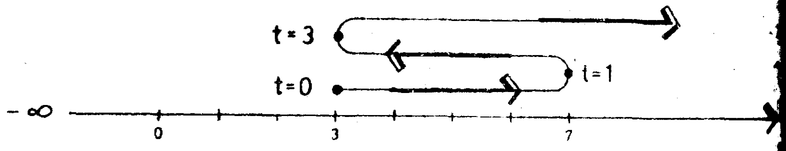


FIG. 3.14

One of the most important types of rectilinear mot that with *constant acceleration*. For example, a freely fallin near the earth's surface moves with a constant acceleratio *freely falling* we mean that air resistance is neglected. The tant of acceleration due to gravity is denoted by g and is n cally equal to 32 ft/sec^2 or 980 cm/sec^2.

Consider a body in rectilinear motion which moves ver upward or downward. The effect of gravity is to slow the down if it is rising and speed it up if it is falling. Suppose a b thrown vertically upward from a point A with an initial ve v_O. It can be shown that its distance s ft from the starting A at the end of t sec is

E (3.7) $\quad s = v_Ot - 16t^2 \quad$ (v_O in f

E (3.8) $\quad s = v_Ot - 490t^2 \quad$ (v_O in cm

Note that $s > 0$ if the body is above A and $s < 0$ if it is bel If the body were thrown downward, then we consider v_O

EXAMPLE 2: A body is thrown vertically upward fro ground with an initial velocity of 96 ft/sec. the maximum height attained by the body.

Solution: Substituting $v_O = 96$ in E (3.7), we have

$$s = 96t - 16t^2 \qquad (1)$$

Then by E (3.5),

$$v = 96 - 32t \qquad (2)$$

At the highest point, $v = O$. Hence from (2)

$$O = 96 - 32t$$
$$t = 3 \text{ sec.}$$

This is the time required to reach the highest point. Substituting $t = 3$ in (1), we get

$$s = 144 \text{ ft.}$$

Hence the maximum height attained by the body is 144 ft.

EXERCISE 3.8

In each of the following, s (in ft) is the directed distance of a moving body or particle from the origin at time t (in sec) on a coordinate line. Describe and diagram the motion for $t \geq 0$.

1. $s = 2t^3 - 15t^2 + 36t$
2. $s = t^3 - 9t^2 + 24t + 3$
3. $s = t^3 - 12t^2 + 5$
4. $s = t^3 - 9t^2 + 15t + 4$

Find the values of t for which the velocity is increasing.

5. $s = t^3 - 6t^2 + 4$
6. $s = t^3 - 12t^2 + 5$
7. $s = t^4 - 8t^3 + 5$
8. $s = (t - 3)^4$
9. If $s = \sqrt{8t} + \sqrt[3]{4t}$, find the velocity and acceleration when $t = 2$.
10. If $s = t^3 - t^2$, find the velocity when the acceleration is 2.
11. If $s = 3t^2 - 16t^{-2}$, when will the acceleration be zero?
12. An object is thrown vertically upward from a point on the ground with an initial velocity of 128 ft/sec. Find (a) its velocity at the end of 3 sec., (b) the time required to reach the highest point, and (c) the maximum height attained.

13. A body is thrown vertically upward from a point on the ground. If it attains a maximum height of 400 meters, find its initial velocity.
14. From the top of a building 42 meters high, a body is thrown vertically upward with an initial velocity of 36 meters per second. Find (a) its greatest distance from the ground and (b) its velocity when it strikes the ground.
15. An object thrown vertically upward from the ground reaches a certain height after 2 sec and returns to the same height on descent, 8 sec later. Find its initial velocity and the height in question.

chapter

4

Differentiation of Transcendental Functions

We shall discuss in this chapter the differentiation of a new class of functions. These functions which are not algebraic are called *transcendental functions*. The trigonometric functions and their inverses, together with the logarithmic and exponential functions, are the simplest transcendental functions.

4.1 The Function

Consider the function f defined by the equation

$$f(u) = \frac{\sin u}{u}.$$

This function assumes the meaningless form $\frac{0}{0}$ for u = 0. However, the limit of this function exists when u approaches zero. To prove this, consider Fig. 4.1 where arc AC subtends an angle u (measured in radians) at the center 0 of a circle of radius r.

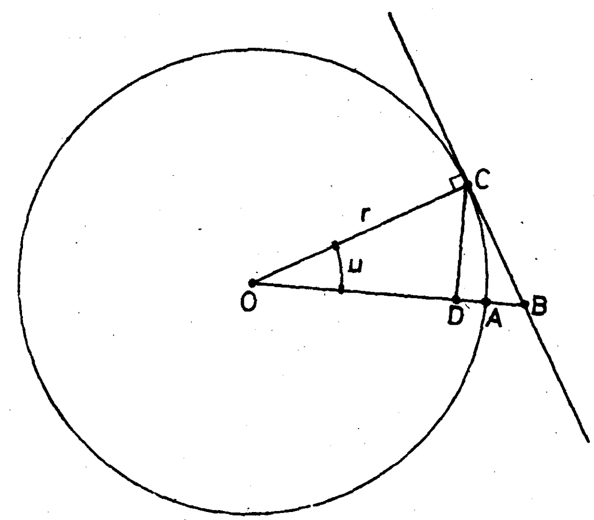


FIG. 4.1

be the perpendicular to OA and BC be the tangent to the circle at C. From the figure, we note that

Area of $\Delta ODC <$ Area of sector AOC $<$ Area of ΔOBC

or $\frac{1}{2}$ (OD) (DC) $< \frac{1}{2}(OA)^2$ (u) $< \frac{1}{2}$ (OC) (CB) (1)

By Trigonometry and with r = OC, we get the following relations:

$$OD = r\cos u$$
$$DC = r\sin u$$
$$CB = r\tan u = r.\frac{\sin u}{\cos u}$$

Substituting these values in (1), we have

$$\frac{1}{2}r^2 \cos u \ \sin u < \frac{1}{2}r^2 \ u < \frac{1}{2}r^2 \frac{\sin u}{\cos u} \qquad (2)$$

Dividing each term of (2) by $\frac{1}{2}r^2 \sin u$, we get

$$\cos u < \frac{u}{\sin u} < \frac{1}{\cos u} \qquad (3)$$

Taking the reciprocals of the terms in (3)

$$\frac{1}{\cos u} > \frac{\sin u}{u} > \cos u \qquad (4)$$

From (4), we note that as $u \to 0$, $\cos u \to 1$ and $\frac{1}{\cos u} \to 1$. Since $\frac{\sin u}{u}$ lies between $\cos u$ and $\frac{1}{\cos u}$, both of which approach one as u approaches zero, then $\frac{\sin u}{u} \to 1$. We now formally state this fact as a theorem.

T (4.1) If the angle u is in radians, then the ratio $\frac{\sin u}{u}$ approaches unity as u approches zero. In symbol,

L10 $$\lim_{u \to 0} \frac{\sin u}{u} = 1$$

EXAMPLE 1: Evaluate $\lim_{x \to 0} \frac{x + \sin x}{x}$

Solution: $$\lim_{x \to 0} \frac{x + \sin x}{x} = \lim \left(1 + \frac{\sin x}{x}\right)$$
$$= \lim_{x \to 0} 1 + \lim_{x \to 0} \frac{\sin x}{x}$$
$$= 1 + 1$$
$$= 2$$

...PLE 2. $\lim_{x \to 0} \frac{\sin^2 3x \cos x}{x}$

Solution:

$$\lim \frac{\sin^2 3x \cos x}{x^2} = \lim_{x \to 0} \frac{\sin^2 3x \cos x}{x^2} \cdot \frac{9}{9}$$
$$= \lim_{x \to 0} \left(\frac{\sin 3x}{3x} \cdot \frac{\sin 3x}{3x} 9 \cos x\right)$$
$$= \lim_{x \to 0} \frac{\sin 3x}{3x} \lim_{x \to 0} \frac{\sin 3x}{3x} \lim_{x \to 0} 9\cos x$$
$$= 1 \cdot 1 \cdot 9 \cdot 1$$
$$= 9$$

EXERCISE 4.1

...each of the following limits:

1. $\lim_{x \to 0} \frac{\sin 3x}{\sin 2x}$
2. $\lim_{x \to 0} \frac{\tan 4x}{3x}$
3. $\lim_{x \to 0} \frac{1 - \cos x}{4x^2}$
6. $\lim_{x \to 0} \frac{\cos x \tan 2x}{\sin 2x}$
7. $\lim_{x \to 0} \frac{\sin 5x}{3x}$
8. $\lim_{x \to 0} \frac{5 - \cos x}{4x^2}$

4. $\lim\limits_{x \to 0} \dfrac{4x^2}{1 - \cos^2 \frac{x}{2}}$

5. $\lim\limits_{x \to 0} \dfrac{x^2 + 4x}{\sin 2x}$

9. $\lim\limits_{x \to 0} \dfrac{1 - \cos 4x}{x \sin 4x}$

10. $\lim\limits_{x \to 0} \dfrac{1 - \cos^2 x}{x^2 \cos x}$

4.2 Differentiation of Trigonometric Functions

The following formulas are used for differentiating metric functions. The symbol u denotes an arbitrary differ function of x.

D12: $\dfrac{d}{dx}(\sin u) = \cos u \dfrac{du}{dx}$

D13: $\dfrac{d}{dx}(\cos u) = -\sin u \dfrac{du}{dx}$

D14: $\dfrac{d}{dx}(\tan u) = \sec^2 u \dfrac{du}{dx}$

D15: $\dfrac{d}{dx}(\cot u) = -\csc^2 u \dfrac{du}{dx}$

D16: $\dfrac{d}{dx}(\sec u) = \sec u \tan u \dfrac{du}{dx}$

D17: $\dfrac{d}{dx}(\csc u) = -\csc u \cot u \dfrac{du}{dx}$

We shall give the proofs of the first three formulas. The of the remaining three should be carried through by the In proving D12, we shall use Definition 2.1 (Chapter 2) proof.

Proof of D12:

Let $y = \sin u$ where u is a function of x. Then we h $\sin(u + \Delta u) - \sin u$. By Definition 2.1,

$$\frac{dy}{du} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\sin(u + \Delta u) - \sin u}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{2\cos(u + \frac{1}{2}\Delta u) \sin \frac{1}{2}\Delta u}{\Delta u} \qquad \text{(why?)}$$

$$= \lim_{\Delta u \to 0} \frac{\cos(u + \frac{1}{2}\Delta u) \sin \frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}$$

$$= \lim_{\Delta u \to 0} \cos(u + \tfrac{1}{2}\Delta u) \lim_{\frac{1}{2}\Delta u \to 0} \frac{\sin \frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}$$

$$= \cos(u + 0) \cdot 1$$

refore,

$$\frac{dy}{du} = \cos u$$

ltiplying both sides by $\dfrac{du}{dx}$, we get

$$\frac{dy}{du} \cdot \frac{du}{dx} = \cos u \frac{du}{dx}$$

$$\frac{dy}{dx} = \cos u \frac{du}{dx}$$

e $y = \sin u$, then

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

of D13:

prove D13, we may use again Definition 2.1 and L10 but , its proof is based on the result already achieved for , by using D12.

In trigonometry, we have the following relations:

$$\sin u = \cos\left(\tfrac{1}{2}\pi - u\right) \quad (1)$$

$$\cos u = \sin\left(\tfrac{1}{2}\pi - u\right) \quad (2)$$

Differentiating (2) with respect to x,

$$\frac{d}{dx}(\cos u) = \frac{d}{dx}\sin\left(\tfrac{1}{2}\pi - u\right)$$

$$= \cos\left(\tfrac{1}{2}\pi - u\right)\frac{d}{dx}\left(\tfrac{1}{2}\pi - u\right)$$

$$= \sin u\left(-\frac{du}{dx}\right)$$

Therefore,

$$\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$$

Proof of D14:

We shall use D12 and D13 to prove D14. Since

$$\tan u = \frac{\sin u}{\cos u}$$

Then differentiating both sides

$$\frac{d}{dx}(\tan u) = \frac{d}{dx}\left(\frac{\sin u}{\cos u}\right)$$

$$= \frac{\cos u\frac{d}{dx}(\sin u) - \sin u\frac{d}{dx}(\cos u)}{\cos^2 u}$$

$$= \frac{\cos u\cos u\frac{du}{dx} - \sin u(-\sin u)\frac{du}{dx}}{\cos^2 u}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\frac{du}{dx}$$

$$\frac{1}{\cos^2 u}\frac{du}{dx}$$

Therefore, we have

$$\frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx}$$

The following examples illustrate the use of the formula for differentiating trigonometric functions:

EXAMPLE 1: Find $\frac{dy}{dx}$ if $y = \sin 4x$

Solution: We note that $y = \sin 4x$ takes the form $y = \sin u$ with $u = 4x$. Hence

$$\frac{dy}{dx} = \cos 4x\frac{d}{dx}(4x) \qquad \text{by D12}$$

$$= \cos 4x\,(4)$$

$$= 4\cos 4x$$

EXAMPLE 2: Find $\frac{dy}{dx}$ if $y = \sin^3 4x$

Solution: If we write $y = \sin^3 4x$ as $y = (\sin 4x)^3$, then it takes the form $y = u^n$ with $u = \sin 4x$ and $n = 3$. This suggests the use of D7. Thus

$$\frac{dy}{dx} = 3\sin^2 4x\frac{d}{dx}(\sin 4x) \qquad \text{by D7}$$

$$= 3\sin^2 4x\cos 4x\frac{d}{dx}(4x) \qquad \text{by D12}$$

$$= 3\sin^2 4x\cos 4x\,(4)$$

$$= 12\sin^2 4x\cos 4x$$

As we get more familiar with the formulas and their uses, we may perform some steps mentally and thus shorten our solution. For instance, in Example 2, we may omit some steps given above. Thus in practice, the problem is worked out simply this way:

$$y = \sin^3 4x$$

$$\frac{dy}{dx} = 3\sin^2 4x\cos 4x\ (4)$$

$$= 12\sin^2 4x\cos 4x$$

EXAMPLE 3: Find $\frac{dy}{dx}$ if $y = \tan^4 5x$

Solution:

$$y = \tan^4 5x$$

$$\frac{dy}{dx} = 4\tan^3 5x\sec^2 5x\ (5) \qquad \text{by D7, D14}$$

$$= 20\tan^3 5x\sec^2 5x$$

EXAMPLE 4: Find the height of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 15 cm.

Solution: This can be solved by the method used in Chapter 3. In fact this is Problem 4 in Exercise 3.6. This time, we shall solve it by using trigonometric functions. In Fig. 4.2, we have

h = height of the cylinder

r = radius of the base

Let V = volume of the cylinder. We are asked to find h for maximum V.

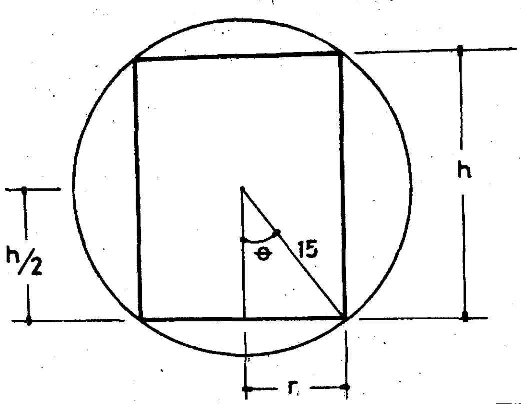


FIG. 4.2

The volume of the cylinder is

$$V = \pi r^2 h \qquad (1)$$

Since we are going to solve this by use of trigonometric functions, we introduce θ as our new variable. From the right triangle in Fig. 4.2, we obtain the following relations:

$$h = 30\cos\theta \qquad (2)$$

$$r = 15\sin\theta \qquad (3)$$

Substituting (2) and (3) in (1) and simpliying,

$$V = 6750\pi \sin^2\theta\cos\theta \qquad (4)$$

Differentiating (4) with respect to θ,

$$\frac{dV}{d\theta} = 6750\pi\left[\sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta)\right]$$

$$= 6750\pi\ (\sin\theta)(2\cos^2\theta - \sin^2\theta)$$

Setting $\frac{dV}{d\theta} = 0$,

$$6750\pi\ (\sin\theta)(2\cos^2\theta - \sin^2\theta) = 0$$

Then we have

$$\sin\theta = 0 \qquad \text{(discard this value)}$$

and

$$2\cos^2\theta - \sin^2\theta = 0$$

$$2\cos^2\theta - (1 - \cos^2\theta) = 0$$

$$3\cos^2\theta - 1 = 0$$

$$\cos\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Substituting this value in (2), we obtain

$$h = 10\sqrt{3}\ \text{cm.}$$

EXERCISE 4.2

Find $\frac{dy}{dx}$ and simplify the result whenever possible

1. $y = \frac{1}{2}x + \frac{1}{4}\sin 2x$
2. $y = \sin 5x - \frac{1}{3}\sin^3 5x$
3. $y = \sin^2 4x + \frac{1}{2}\cos 8x$
4. $y = 3x\cos\frac{x}{3} - 9\sin\frac{x}{3}$
5. $y = \frac{3}{8}x + \frac{3}{8}\sin x\cos x + \frac{1}{4}\cos^3 x\sin x$
6. $y = x^2\sin x + 2x\cos x - 2\sin x$
7. $y = \sin(x+4)\cos(x-4)$
8. $y = \frac{1-\cos 4x}{\sin 4x}$
9. $y = 3\tan 2x + \tan^3 2x$
10. $y = \sec^2 4x + \tan^2 4x$
11. $y = \csc x - \frac{1}{3}\csc^3 x$
12. $y = \sec^4 x - 2\tan^2 x$
13. $y = \sec^3 2x - 3\sec 2x$
14. $y = \csc^4 x - 2\cot^2 x$
15. $y = -\frac{3}{5}\cot^5\frac{x}{3} + \cot^3\frac{x}{3} - 3\cot\frac{x}{3} - x$
16. $\cos(xy) = x - y$
17. $\sin(x+y) = x + y$
18. $x\cos y = \sin(x+y)$

... $\frac{x}{y} = \frac{y}{x}$

... $y + y\cos x = 0$

... the following problems by making use of Trigonometric ...

... the dimensions of the right circular cylinder of maxi... lateral surface area which can be inscribed in a sphere ... radius 4 in.

... strength of a rectangular beam is proportional to the ...dth and the square of the depth. Find the dimensions ... the strongest beam that can be cut from a circular log of ...ius R.

... the length of the shortest ladder which will reach from ... ground level to a high vertical wall if it must clear an 8-ft ...ical fence which is 27 ft from the wall.

... the volume of the largest conical tent that can be cons...ted with a slant height of 12 ft.

... the area of the largest regular cross that can be ins...ed in a circle of radius R. (A regular cross is a square ...mounted by four equal rectangles.)

... ladder 10 ft long leans against a vertical wall. The upper ... slips down the wall at 5 ft/sec. How fast is the ladder ...ing when it takes an angle of 30° with the ground?

... of the equal sides of an isosceles triangle has cons... length of 4 ft. If the angle θ between these sides in...es at the rate of 10 rad/sec, find the rate at which the ... is increasing when $\theta = \frac{\pi}{3}$

... hypotenuse of a right triangle is 25 ft. If one of the acute ...les increases at the rate of 4 degrees per second, how ... is the area increasing when the angle is 30 degrees?

...rentiation of Inverse Trigonometric Functions

... recall from trigonometry that

... Arcsinx iff $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Note that without restricting the values of y in th $[-\frac{\pi}{2}, \frac{\pi}{2}]$ the equation y = Arcsinx does not define a

The reason for this is that for any value of x in the interv there are infinitely many values of y which will satisfy tion y = Arcsinx*. However, with this restriction, we for each value of x in $[-1, 1]$, there is a unique value instance,

$$y = \text{Arcsin}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$y = \text{Arcsin}(-1) = -\frac{\pi}{2}$$

The notation $\text{Sin}^{-1}x$ is often used for Arcsinx but in t we shall use the "Arc" notation. * The definition of the re inverse trigonometric functions are as follow:

$y = \text{Arccos}x$ iff $x = \cos y$ and $0 \leq y \leq \pi$

$y = \text{Arctan}x$ iff $x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$y = \text{Arccot}x$ iff $x = \cot y$ and $0 < y < \pi$

$y = \text{Arcsec}x$ iff $x = \sec y$ and $-\pi \leq y < -\frac{\pi}{2}$ for $x \leq -$

$0 \leq y < \frac{\pi}{2}$ for $x \geq 1$

$y = \text{Arccsc}x$ iff $x = \csc y$ and $-\pi < y \leq -\frac{\pi}{2}$ for $x \leq -$

$0 < y \leq \frac{\pi}{2}$ for $x \geq 1$

The following formulas are used for differentiating trigonometric functions. The symbol u denotes an arbit ferentiable function of x.

* The student who is not so familiar with the properties of unverse trig functions should refer to any standard text on trigonometry for review.

** The notation $\text{Sin}^{-1}x$ is considered inconvenient by some people since it read as "sinx with exponent -1" The -1 in this expression is not an expone fore $\text{Sin}^{-1}x$ does not mean $(\sin x)^{-1}$ or $\frac{1}{\sin x}$

$$\frac{d}{dx}(\text{Arcsin}u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}(\text{Arccos}u) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}(\text{Arctan}u) = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}(\text{Arccot}u) = \frac{-1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}(\text{Arcsec}u) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$$

$$\frac{d}{dx}(\text{Arccsc}u) = \frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx}$$

ofs:

Let $y = \text{Arcsin}u$ (1)

Then $u = \sin y$ (2)

Differentiating (2) with respect to x

$$\frac{du}{dx} = \cos y \frac{dy}{dx} \qquad (3)$$

Solving (3) for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{\cos y}\frac{du}{dx} \qquad (4)$$

But $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1-u^2}$ The positive sign of the radical is chosen since $\cos y > 0$ for $0 \leq y \leq \frac{\pi}{2}$). Hence (4) becomes

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

Substituting (1) in (5)

$$\frac{d}{dx}(\text{Arcsin}u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

Proof of D22:

Let $y = \text{Arcsec}\,u$ (1

Then $u = \sec y$ (2

Differentiating (2) with respect to x

$$\frac{du}{dx} = \sec y \tan y \frac{dy}{dx}$$ (

Solving (3) for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}\frac{du}{dx}$$

But $\sec y = u$ and $\tan y = \sqrt{\sec^2 y - 1} = \sqrt{u^2 -}$ positive sign of the radical is chosen since tany $-\pi \leq y < -\frac{\pi}{2}$ and $0 \leq y < \frac{\pi}{2}$). Substituti values in (4).

$$\frac{dy}{dx} = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$$

Since $y = \text{Arcsec}\,u$, we finally get

$$\frac{d}{dx}(\text{Arcsec}\,u) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$$

The student is urged to give the proofs of the rema mulas. Here are some examples to illustrate the use of th las above.

EXAMPLE 1: Find $\frac{dy}{dx}$ if $y = \text{Arcsin}\,3x$

Solution: Since $y = \text{Arcsin}\,3x$ takes the form y = where $u = 3x$, then we use D18. Thus

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x)^2}}\frac{d}{dx}(3x)$$

$$= \frac{3}{\sqrt{1-9x^2}}$$

EXAMPLE 2. If $y = \text{Arctan}\,\frac{x}{4}$, find $\frac{dy}{dx}$.

Solution: By D20, we have

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{x}{4}\right)^2}\frac{d}{dx}\left(\frac{x}{4}\right)$$

$$= \frac{1}{1+\frac{x^2}{16}}\left(\frac{1}{4}\right)$$

$$= \frac{4}{16+x^2}$$

EXAMPLE 3. A ladder 25 ft long leans against a vertical wall. If the lower end is pulled away at the rate of 6 ft/sec, how fast is the angle between the ladder and the floor changing when the lower end is 7 ft from the wall?

Solution: In Fig. 4.3, we let x = distance of the lower end of the ladder AB from the wall CB and let θ = angle between the ladder and the floor CA. We want to find $\frac{d\theta}{dt}$ when $\frac{dx}{dt} = 6$ ft/sec and $x = 7$.

Since $\cos\theta = \frac{x}{25}$

then $\theta = \text{Arccos}\,\frac{x}{25}$

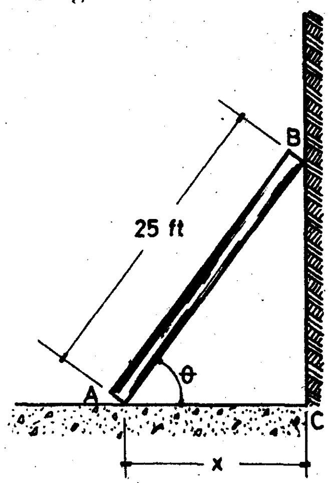
Differentiating with respect to t,

$$\frac{d\theta}{dt} = \frac{-1}{\sqrt{1-\frac{x^2}{625}}}\;\frac{1}{25}\;\frac{dx}{dt}$$

Substituting $x = 7$ and $\frac{dx}{dt} = 6$, we get

$$\frac{d\theta}{dt} = -\frac{1}{4}\ \text{rad/sec.}$$

The minus sign indicates that θ is decreasing.



EXERCISE 4.3

Find $\frac{dy}{dx}$ and simplify the result whenever possible

1. $y = \text{Arcsin}\sqrt{1-x^2}$
2. $y = \text{Arccos}\,\frac{x}{1-x}$
3. $y = \text{Arctan}\,\frac{4}{x}$
4. $y = \text{Arccot}\,(\tan 2x)$
5. $y = \text{Arcsec}\sqrt{4x+1}$

$y = \text{Arccsc}\,\frac{x}{2}$

$y = \frac{1}{4}\,\text{Arctan}\,\frac{4\sin x}{3+5\cos x}$

$y = \text{Arccos}\,\frac{x}{2} - \frac{\sqrt{4-x^2}}{x}$

$y = x\sqrt{1-4x^2} + \frac{1}{2}\,\text{Arcsin}\,2x$

$y = \sqrt{x^2-4} - 2\,\text{Arcsec}\,\frac{x}{2}$

$y = \text{Arccot}\,x + \text{Arctan}\,\frac{2+x}{1-2x}$

$y = \text{Arctan}\,x + \text{Arcsec}\sqrt{1+x^2}$

$y = x\,\text{Arcsin}^2 x - 2x + 2\sqrt{1-x^2}\ \text{Arcsin}\,x$

$y = (x-1)\sqrt{2x-x^2} - \text{Arccos}\,(x-1)$

$\text{Arcsin}\,\frac{x}{y} + \text{Arccos}\,\frac{y}{x} = 1$

$\text{Arctan}\,\frac{x}{y} = x - y$

$y = \sqrt{x^2-a^2} + a\,\text{Arccot}\left(\frac{\sqrt{x^2-a^2}}{a}\right)$

$y = \sqrt{a^2-x^2} + a\,\text{Arcsin}\,\frac{x}{a}$

$y = ab\ \text{Arctan}\left(\frac{a\cot x}{b}\right)$

$y = \text{Arccot}\,\frac{x+b}{1-bx}$

Solve the following problems by making use of inverse trigonometric functions.

The lower edge of a picture is 4 ft, the upper edge 9 ft above the eye of an observer. At what horizontal distance should he stand if the angle subtended by the picture is a maximum?

22. At what point on the line $x = 4$ does the line segment from $(0, 0)$ to $(0, 6)$ subtend the greatest angle?
23. A searchlight, 1/2 mi from a straight shore, rotates at the rate of 2 rev/min. How fast is the spot of light from the searchlight moving along the shore when it is 1 mi from the point on the shore nearest the searchlight?
24. An isosceles triangle has legs 10 cm. The base decreases at the rate of 4 cm/sec. Find the rate of change of the angle at the apex when the base is 16 cm.
25. A ladder 14 ft long is leaning against a fence 8 ft high with the upper end projecting over the fence. If the lower end slides away from the fence at the rate of 2 ft/ sec, find the rate at which the angle between the ladder and the ground is changing when the upper end is just at the top of the fence.
26. A searchlight is trained on an object falling under the influence of gravity from a height of 500 ft. Find the rate at which the beam of light is following the object when the object is 100 ft from the ground. Assume that the searchlight is 200 ft from the point where the object hits the ground.

4.4 The functions $(1 + u)^{\frac{1}{u}}$

The function defined by the equation

$$y = (1 + x)^{\frac{1}{x}}$$

assumes the meaningless form 1^{∞} for $x = 0$. However, it can be shown that the limit of this function exists when x approaches zero.* This limit is denoted by e. That is,

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$$

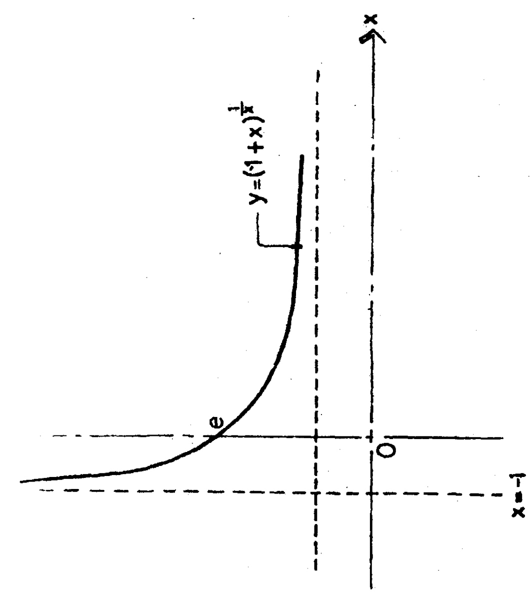
*A rigorous proof showing the existence of this limit is beyond the scope of this book.

...number e is a nonterminating and nonrepeating decimal and ...lue can be obtained to any desired accuracy. In practice, the ...imate value assigned to e is 2.718.**

The graph of $y = (1 + x)^{\frac{1}{x}}$ is shown in Fig. 4.4. It shows ...trically that as x approaches zero from the left, y decrea... approaches e as a limit. On the other hand, as x approa... zero from the right, y increases and likewise approaches e ... limit. Furthermore, we note that as x becomes positively ..., y approaches 1 as a limit and as x approaches -1 from ...ht, y increases without bound. Hence $y = 1$ and $x = -1$ are ...totes of the curve.

In general, if u is a function of x, we define e as

$$\text{L11:} \qquad e = \lim_{u \to 0} (1 + u)^{1/u}$$



**...his value can be computed by the infinite series method which is not discussed ...book.

Since $y = \log_b u$, then we have

$$\frac{d}{dx}(\log_b u) = \frac{1}{u}(\log_b e)\frac{du}{dx}$$

D25 can be obtained directly from D24. Note that if we replace b by e in D24, we get

$$\frac{d}{dx}(\log_e u) = \frac{1}{u}(\log_e e)\frac{du}{dx}$$

But $\log_e u = 1nu$ and $\log_e e = 1n\,e = 1$ by P4. The equation above finally becomes D25, that is

$$\frac{d}{dx}(1nu) = \frac{1}{u}\frac{du}{dx}$$

EXAMPLE 1. Find $\frac{dy}{dx}$ if $y = \log_5 (4x + 3)$.

Solution: Let $u = 4x + 3$. Then by D24.

$$\frac{dy}{dx} = \frac{1}{4x+3}(\log_5 e)(4)$$

$$= \frac{4(\log_5 e)}{4x + 3}$$

EXAMPLE 2: Find $\frac{dy}{dx}$ if $y = 1n\,(2x + 1)^4$

Solution: Let $u = (2x + 1)^4$. Then by D25,

$$\frac{dy}{dx} = \frac{1}{(2x+1)^4}\,4(2x+1)^3\;2$$

$$= \frac{8}{2x + 1}$$

Alternative Solution: Another solution is to apply first P3 and then use D25. Thus

$$y = 1n\,(2x + 1)^4$$

$$= 41n\,(2x + 1)$$

$$\frac{dy}{dx} = 4\cdot\frac{1}{2x+1}(2)$$

$$= \frac{8}{2x+1}$$

EXAMPLE 3: If $y = 1n\sqrt{\frac{x+4}{x-4}}$, find $\frac{dy}{dx}$.

Solution: This can be solved by direct application of D25. But we shall solve this by applying first R2, P3 and P2 before using D25. Thus

$$y = 1n\sqrt{\frac{x+4}{x-4}}$$

$$= 1n\left(\frac{x+4}{x-4}\right)^{\frac{1}{2}} \qquad \text{by R2}$$

$$= \tfrac{1}{2}\,1n\left(\frac{x+4}{x-4}\right) \qquad \text{by P3}$$

$$= \tfrac{1}{2}\left[1n\,(x+4) - 1n\,(x-4)\right] \qquad \text{by P2}$$

$$\frac{dy}{dx} = \tfrac{1}{2}\left[\frac{1}{x+4} - \frac{1}{x+4}\right]$$

$$= \frac{-4}{x^2 - 16}$$

EXERCISE 4.4

Find $\frac{dy}{dx}$ and simplify whenever possible.

1. $y = \log\sqrt{2x+5}$
2. $y = \log \sin^2 4x$

3. $y = \log \sqrt[3]{\sqrt{12x}}$

4. $y = \log \left[(x-1)^3 \ (x+2)^4\right]$

5. $y = \ln (x+3)^4$

6. $y = \ln (x + \sqrt{x^2+1})$

7. $y = \ln^4 (x+3)$

8. $y = \ln \sqrt{\dfrac{a+bx}{a-bx}}$

9. $y = \ln \left(\dfrac{1-\sin x}{1+\sin x}\right)^3$

10. $y = x\text{Arctan}x - \ln \sqrt{1+x^2}$

11. $y = \ln(\sec x + \tan x)$

12. $y = \ln \dfrac{(x-5)^2}{(x-4)^3}$

13. $y = \ln \dfrac{x^2(x+1)}{(x+2)^3}$

14. $y = x\text{Arcsec}2x - \frac{1}{2} \ln (2x + \sqrt{4x^2-1})$

15. $y = x^4 (1 - \ln x^4)$

16. $y = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2 \ln (x + \sqrt{x^2+a^2})$

17. $y = x\text{Arctan}\dfrac{x}{a} - \dfrac{a}{2} \ln(a^2+x^2)$

18. $y = \ln (\ln \sec x)$

19. $y = \ln(\ln 4x)$

20. $\sin y = \ln(x+y)$

... $y^2) + 2\text{Arctan}\dfrac{y}{x} = 0$

... $y') - xy$

... $y \ln x = 1$

...hmic Differentiation

...section, we shall learn how to find the derivative ...tion which is expressed as a product, quotient, power of two or more differentiable functions of x by a pro... ...own as *logarithmic differentiation*. This procedure ... the following steps:

Take the natural logarithm of both sides of the equa... ...tion which defines the function.

...mplify the right member of the resulting equation by making use of the properties or laws of logarithms.

Differentiate with respect to x and solve for $\dfrac{dy}{dx}$.

... 1 If $y = (2x+1)\sqrt{3x+5}$, find $\dfrac{dy}{dx}$ by logarithmic differentiation.

...ion Taking the logarithm of both sides,

$$\ln y = \ln (2x+1)\sqrt{3x+5}$$

$$= \ln (2x+1) + \tfrac{1}{2} \ln (3x+5) \qquad \text{by P1, R2, P3}$$

...fferentiating with respect to x,

$$\frac{dy}{y\,dx} = \frac{1}{2x+1}(2) + \frac{1}{2}\,\frac{1}{3x+5}(3) \qquad \text{by D25}$$

$$\frac{dy}{dx} = y\left[\frac{2}{2x+1} + \frac{3}{2(3x+5)}\right]$$

$$= \frac{(2x+1)\sqrt{3x+5}\,[4(3x+5)+3(2x+1)]}{2(2x+1)(3x+5)}$$

Then

$$\ln y = u \ln a$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \ln a \frac{du}{dx}$$

$$\frac{dy}{dx} = y\,(\ln a)\frac{du}{dx}$$

Replacing y by a^u, we obtain D27, that is

$$\frac{d}{dx}(a^u) = a^u\,(\ln a)\frac{du}{dx}$$

Note: D28 may be obtained directly from D
by replacing a by e.

EXAMPLE 1. If $y = 4^{2x}$, find $\frac{dy}{dx}$.

Solution:

$$y = 4^{2x}$$

$$\frac{dy}{dx} = 4^{2x}\,(\ln 4)\,(2) \quad \text{by D}$$

$$= 4^{2x}\,(2\ln 4)$$

$$= 4^{2x}\,(\ln 16)$$

EXAMPLE 2. If $y = e^{\sin x}$, find $\frac{dy}{dx}$

Solution:

$$y = e^{\sin x}$$

$$\frac{dy}{dx} = e^{\sin x}\,(\cos x) \quad \text{by D}$$

EXAMPLE 4.6

Find $\frac{dy}{dx}$ and simplify whenever possible.

1. $y = 3^{4x}$
2. $y = \dfrac{1 + 2^x}{1 - 2^x}$
3. $y = 4^x \ln 4x$
4. $y = e^{-4x}$
5. $y = e^x e^{\ln x}$
6. $y = \ln \dfrac{e^{2x} - 1}{e^{2x} + 1}$
7. $y = \ln (e^x x^2)$
8. $y = x^{2^x}$
9. $y = x^{e^x}$
10. $3^x + 3^y = 6$
11. $x^y + 2^y = 8$
12. $e^{xy} + \ln (xy) = 3$
13. $e^x + e^y = e^x e^y$
14. $e^y \sin x = \dfrac{y}{x}$
15. $e^{\ln 4x} + e^{\ln 4y} = 1$
16. $e^{x + y} = \ln \dfrac{x}{y}$
17. $x^y + e^x = a$

18. Find the value of A so that $y = Ae^{2t}$ will satisfy the equation $y'' - 2y' - 3y = e^{2t}$.

19. Find the minimum value of $y = 4e^{x} + 9e^{-x}$.

20. Find the maximum point of the graph of $y = e^{-x^2}$

21. Find the area of the largest triangle cut from the first quadrant by a line tangent to $y = e^{-2x}$

4.9 The Hyperbolic Functions

Certain combinations of the exponential function e^x and e^{-x} occur frequently in mathematics, science and engineering. These functions are called *hyperbolic functions**. They are defined as follows:

DEFINITION 4.1 $\quad \sinh x = \dfrac{e^x - e^{-x}}{2}$

DEFINITION 4.2 $\quad \cosh x = \dfrac{e^x + e^{-x}}{2}$

DEFINITION 4.3 $\quad \tanh x = \dfrac{\sinh x}{\cosh x}$

DEFINITION 4.4 $\quad \coth x = \dfrac{\cosh x}{\sinh x}$

DEFINITION 4.5 $\quad \text{sech}\, x = \dfrac{1}{\cosh x}$

DEFINITION 4.6 $\quad \text{csch}\, x = \dfrac{1}{\sinh x}$

The notation *sinhx* is read "hyperbolic sine of x". The others are read in the same manner.

The following identities can be deduced directly from the definitions of the hyperbolic functions.

H1. $\quad \cosh^2 x - \sinh^2 x = 1$

*They are called hyperbolic functions because they can be related to a hyperbola. Recall that the trigonometric functions are also called circular functions because of their relation to a circle.

H2. $\quad \tanh^2 x + \text{sech}^2 x = 1$

H3. $\quad \coth^2 x - \text{csch}^2 x = 1$

H4. $\quad \sinh 2x = 2\sinh x\cosh x$

H5. $\quad \cosh 2x = \cosh^2 x + \sinh^2 x$

$$= 1 + 2\sinh^2 x$$
$$= 2\cosh^2 x - 1$$

...PLE 1. Prove that $\cosh^2 x - \sinh^2 x = 1$.

Proof: Since by definitions 4.1 and 4.2

$(\sinh x) = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Then, we have

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}$$

$$= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4}$$

$$= \frac{2 + 2}{4}$$

$$= 1$$

...PLE 2. Prove that $\sinh(-x) = -\sinh x$

Proof: By Definition 4.1

$$\sinh(-x) = \frac{e^{(-x)} - e^{-(-x)}}{2}$$

Finally substituting (1) and (6) in (4), we get the desired form

$$\frac{d}{dx}(\sinh^{-1}u) = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$$

Alternative proof of D35:

Let u be a differentiable function of x. Then by Defi tion 4.8,

$$\sinh^{-1}u = \ln(u + \sqrt{u^2+1})$$

Differentiating with respect to x

$$\frac{d}{dx}(\sinh^{-1}u) = \frac{1}{u+\sqrt{u^2+1}}\left(1 + \frac{u}{\sqrt{u^2+1}}\right)\frac{du}{dx}$$

$$= \frac{1}{u+\sqrt{u^2+1}}\left(\frac{\sqrt{u^2+1}+u}{\sqrt{u^2+1}}\right)\frac{du}{dx}$$

$$= \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$$

EXAMPLE 1. If $y = \sinh^{-1}4x$, find $\frac{dy}{dx}$.

Solution: $$\frac{dy}{dx} = \frac{1}{\sqrt{(4x)^2+1}}\frac{d}{dx}(4x)$$

$$= \frac{4}{\sqrt{16x^2+1}}$$

EXAMPLE 2. If $y = \cosh^{-1}(2x-1)$, find $\frac{dy}{dx}$.

Solution: $$\frac{dy}{dx} = \frac{1}{\sqrt{(2x-1)^2-1}}\frac{d}{dx}(2x-1)$$

$$= \frac{1}{\sqrt{x^2-x}}$$

EXERCISE 4.9

y′ and simplify whenever possible.

sinh $^{-1}\sqrt{x^2-1}$

cosh $^{-1}\frac{x}{4}$

tanh $^{-1}(1-2x)$

sech $^{-1}(x+1)$

coth $^{-1}(\cosh x)$

tanh $^{-1}\left(\frac{1-x}{1+x}\right)$

cosh $^{-1}(\sec 2x)$

csch $^{-1}(\tan x)$

ln $\left(\sqrt{x^2+1}+x\right) - \sinh^{-1}x$

ln $(1-9x^2) + 2\tanh^{-1}3x$

The Indeterminate Forms

this chapter, we shall study two theorems which have theoretical importance in Calculus. We shall also learn a method for finding the limit of a quotient of two especially when such quotient can not be evaluated by the limit theorem (L6) in Chapter 1.

's Theorem

theorem which we shall state below is a very useful the proof of many theorems in Calculus. This theorem ulated by Michel Rolle (French Mathematician, 1652-

1 ROLLE'S THEOREM

If a function f(x) is continuous in the closed interval [a, b]; if $f'(x)$ exists on the open interval (a, b); and if f(a) = f(b) = 0, then there is a number c in (a, b) such that $f'(c) = 0$.

omit the proof of this theorem in this book. However, of this theorem can be appreciated on the basis of its evidence. Consider the graph of y = f(x) in Fig. 5.1. that between the points A(a, o) and B(b, o), there exists point on the curve where the tangent is horizontal In Fig. 5.1, the point in question is P. However, be more than one number in the open interval (a, b) f'(x) 0. Thus in Fig. 5.2, the tangent line is horizontal and x c_2. That is, $f'(c_1) = f'(c_2) = 0$.

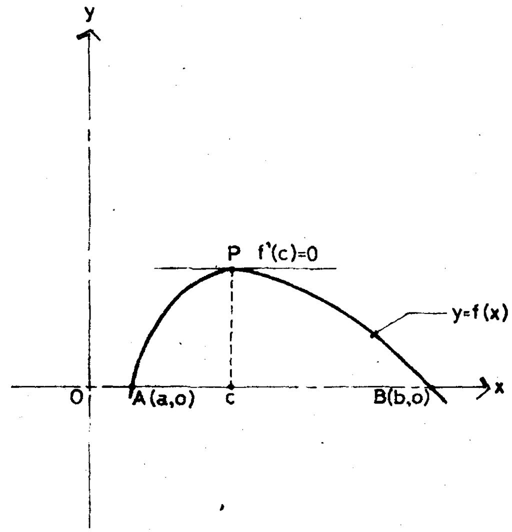


FIG. 5.1

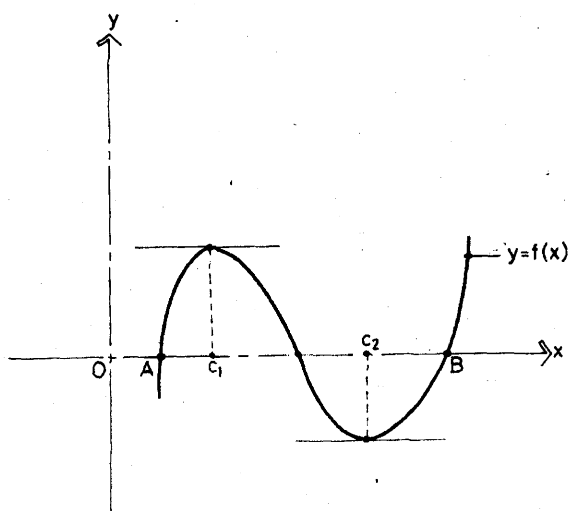


FIG. 5.2

EXAMPLE. Consider the function $f(x) = x^2 -$ the interval [1, 3]. Since f(x) is a polyno it is continuous on [1, 3]. Also f'(x) exists for all x in (1, 3). Finally we note $= f(3) = 0$. Hence the three conditions of thesis of Rolle's thereorem are satisfied

must be a number c in (1, 3) which satisfies the conclusion of the theorem. At $x = c$, $f'(c) = 2c - 4$. But $f'(c) = 0$. Therefore, $2c - 4 = 0$ and we get $c = 2$.

5.2 Mean Value Theorem

The *mean value theorem* (or law of the mean) is one of the important theorems of Calculus. For instance, it is used to estimate the values of functions when direct calculation is difficult. It is also used to prove that two functions having the same derivative must differ by a constant. These are but only two of its important uses.

T(5.2) MEAN VALUE THEOREM

If a function f(x) is continuous on the closed interval [a, b] and if f'(x) exists on the open interval (a, b), then there is a number c in (a, b) such that

$$\text{E(5.1)} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

We shall use Rolle's theorem in the analytic proof of the mean value theorem. Therefore, we must first form a function F(x) which will satisfy the three conditions of Rolle's theorem. Let this function be given by the equation

$$F(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a) - f(x) \qquad (1)$$

Note that F(x) is continuous on [a, b], differentiable and $F(a) = F(b) = 0$. Differentiating (1) with respect to x

$$F'(x) = \frac{f(b) - f(a)}{b - a} - f'(x) \qquad (2)$$

Since F(x) satisfies the three condition of Rolle's theorem, then there exists a number c in (a, b) such that $F'(c) = 0$. Hence, at $x = c$, equation (2) becomes

$$F'(c) = \frac{f(b) - f(a)}{b - a} - f'(c) = 0$$

or equivalently

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

which was to be proved.

The mean value theorem may be interpreted geometricall
n Fig. 5.3, we see that the ratio

$$\frac{f(b) - f(a)}{b - a}$$

is the slope of the line through A $\big((a, f(a)\big)$ and B $\big(b, f(b)\big)$ whi $f'(c)$ is the slope of the tangent of the curve $y = f(x)$ at the poi P $\big(c, f(c)\big)$. Thus the mean value theorem states that between the poi A and B on the curvey $y = f(x)$, there is a point where the tangent line parellel to the line through A and B.

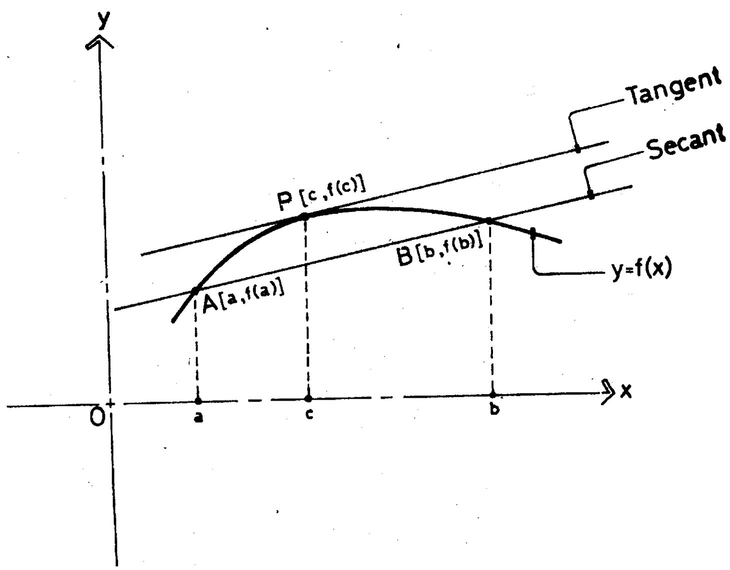


FIG. 5.3

EXAMPLE 1. Given $f(x) = x^2 + 2x - 1$ and $[0, 1]$. Verify that that hypothesis of the mean value theorem is satisfied. Find the value of c that satisfies its conclusion.

Solution: $f(x) = x^2 + 2x - 1$ is continuous on the closed interval $[0, 1]$ since it is a polynomial. We also note that $f'(x) = 2x + 2$ exists on the open interval $(0, 1)$. Hence the hypothesis of the mean value theorem is satisfied. To find the value of c which will satisfy its conclusion, we proceed as follows:

Since $f(x) = x^2 + 2x - 1$, $a=1$ and $b=0$, then

$$f(a) = f(1) = -1$$

$$f(b) = f(0) = 2$$

Since $f'(x) = 2x + 2$, then when $x = c$, we have

$$f'(c) = 2c + 2$$

Substituting these values in E(5.1), we get

$$2c + 2 = \frac{2 + 1}{1 - 0}$$

and $\qquad c = ½$

EXAMPLE 2. Use the mean value theorem to prove that $10.77 < \sqrt{117} < 10.85$.

Solution: Let $f(x) = \sqrt{x}$, $a = 100$ and $b = 117$. Then $f(a) = f(100) = \sqrt{100} = 10$, $f(b) = f(117) = \sqrt{117}$ and since $f'(x) = \dfrac{1}{2\sqrt{x}}$, then $f'(c) = \dfrac{1}{2\sqrt{c}}$ Substituting these values in E(5.1), we have

$$\frac{1}{2\sqrt{c}} = \frac{\sqrt{117} - 10}{117 - 100}$$

Solving for $\sqrt{117}$, we get

$$\sqrt{117} = \frac{17}{2\sqrt{c}} + 10$$

where $100 < c < 117$. Since $(10)^2 = 100$ and $(11)^2 = 121$, then it follows that

$$100 < c < 121$$

or $$10 < \sqrt{c} < 11$$

Taking the reciprocal, we have

$$\frac{1}{10} > \frac{1}{\sqrt{c}} > \frac{1}{11}$$

Multiplying the inequality above by $\frac{17}{2}$ and then adding 10,

$$\frac{17}{20} + 10 > \frac{17}{2\sqrt{c}} + 10 > \frac{17}{22} + 10$$

$$10.85 > \sqrt{117} > 10.77$$

The inequality above is equivalent to

$$10.77 < \sqrt{117} < 10.85$$

which was to be proved.

Now, suppose we write E(5.1) in the form

$$f(b) = f(a) + (b - a)\, f'(c)$$

and then consider again Fig. 5.3 Note that if b is near a, then c is also near a. That is, c comes closer and closer to a as the difference $b - a$ gets smaller and smaller. Then we can see that when $b - a$ is sufficiently small, c approximates the value of a, i.e., $c \doteq a$. The symbol $\doteq$ is read "is approximately equal to". It follows that $f'(c) \doteq f'(a)$. Replacing $f'(c)$ by $f'(a)$ in the equation above, we get

E(5.2) $$f(b) \doteq f(a) + (b - a)\, f'(a)$$

...on may be used to approximate the value of a function ... b when $b - a$ is sufficiently small.

...3. Approximate the value of $\sqrt{82}$.

...ion: Let $f(x) = \sqrt{x}$, $a = 81$ and $b = 82$. Then

$$f(a) = \sqrt{81} = 9$$
$$f(b) = \sqrt{82}$$
$$f'(x) = \frac{1}{2\sqrt{x}}$$
$$f'(a) = \frac{1}{2\sqrt{81}} = \frac{1}{18}$$

Substituting these values in E(5.2), we get

$$\sqrt{82} \doteq 9 + (82 - 81) \cdot \frac{1}{18}$$

$$\doteq \frac{163}{18} \text{ or } 9.06$$

EXERCISE 5.1

... that the three conditions of the hypothesis of Rolle's ... satisfied by the given function on the indicated inter... ... the value of c which satisfies the conclusion of the ...

$f(x) = x^2 - x - 2$, $[-1, 2]$

$f(x) = x^3 - 3x$, $[0, \sqrt{3}]$

$f(x) = x \ln x$, $[0, 1]$

$f(x) = \sin x$, $[0, \pi]$

...the following, find c such that E(5.1) is satisfied.

$f(x) = x^2$, $[3, 4]$

$f(x) = \sqrt{x}$, $[4, 9]$

$f(x) = e^x$, $[0, 1]$

8. $f(x) = \ln x$, $\left[\frac{1}{2}, \frac{3}{2}\right]$

Use the mean value theorem to prove each of the follo

9. $2.11 < \sqrt[3]{9.4} < 2.12$
10. $2.071 < \sqrt{4.3} < 2.075$
11. $0.17 < \ln(1.2) < 0.20$

Explain why the mean value theorem does not apply:

12. $y = x^{2/3}$, $[-1, 1]$
13. $y = \dfrac{4}{x-2}$, $[1, 3]$

Use E(5.2) to approximate the value of each of the fo

14. $\sqrt[5]{33}$
15. $\sqrt{65}$
16. $(2.03)^4$
17. $\sqrt[3]{125.8}$
18. $(9)^{2/3}$
19. $\sqrt[4]{244}$
20. $\dfrac{1}{\sqrt{26}}$

5.3 L' Hospital's Rule

In section 1.4 of Chapter 1, we learned how mit of a quotient of two functions when the num nominator approach zero. In evaluating such a changed it into a form to which the limit theorems c

That is, we employed certain algebraic manipul factoring an expression or rationalizing the deno ever, there are functions having indeterminate fo

not be evaluated by the methods mentioned above. The of this section is to introduce a systematic method for the limits of such functions. This systematic method as *L'Hospital's Rule** and we shall abbreviate this as our convenience. This rule is stated here somewhat , without strictly mentioning the specific condi- ed. The proof is also omitted here.

L'Hospital's Rule

If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the latter limit exists.

rds, LHR states that *to evaluate the limit of the frac- that takes the form $\frac{0}{0}$ at $x = a$, differentiate the numera- nominator separately* and then take the limit of the* $\frac{f'(x)}{g'(x)}$. In case, this new fraction assumes again the the process may be repeated, that is, reapply LHR. How- reapplying LHR at any stage, we may simplify first otient whenever possible. For instance, simplification ation of common factors of the denominator and may be done first before reapplying LHR. The use of illustrated in the next three sections.

terminate Forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$

$f(x) = \lim\limits_{x \to a} g(x) = 0$, then the function defined by

to have the indeterminate form $\frac{0}{0}$ at $x = a$. To

French mathematician G.F.A. L'Hospital (1661-1704), who popular- textbook published in 1696.

evaluate $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$, we apply LHR. Consider the

below

EXAMPLE 1. Evaluate $\lim\limits_{x \to 0} \dfrac{e^{2x} - \cos x}{x}$

Solution: The quotient assumes the form $\dfrac{0}{0}$ when x

Applying LHR, we get

$$\lim_{x \to 0} \frac{e^{2x} - \cos x}{x} = \lim_{x \to 0} \frac{2e^{2x} + \sin x}{1}$$

$$= 2e^{0} + \sin 0$$

$$= 2(1) + 0$$

$$= 2$$

If the $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} g(x) = \infty$, then the fraction d

$\dfrac{f(x)}{g(x)}$ is said to have the indeterminate form $\frac{\infty}{\infty}$ at x = a. LH

applicable for this form.

EXAMPLE 2. Evaluate $\lim\limits_{x \to \infty} \dfrac{x^2}{e^{2x}}$

Solution: Since $\lim\limits_{x \to \infty} \dfrac{x^2}{e^{2x}} = \dfrac{\infty}{\infty}$, we apply LHR. T

$$\lim_{x \to \infty} \frac{x^2}{e^{2x}} = \lim_{x \to \infty} \frac{2x}{2e^{2x}} \qquad (\text{}$$

$$= \lim_{x \to \infty} \frac{x}{e} \qquad (\text{cancelli}$$

$$= \lim_{x \to \infty} \frac{1}{2e^{2x}} \qquad (\text{reapplyi}$$

$$= 0$$

EXERCISE 5.2

Evaluate each of the following limits:

1. $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x - \sin x}$

2. $\lim\limits_{x \to 1} \dfrac{x \ln x - x^2 + x}{(x-1)^2}$

3. $\lim\limits_{x \to 0} \dfrac{(1 - e^x)^2}{x \sin x}$

4. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x^2}$

5. $\lim\limits_{x \to \pi} \dfrac{\ln \cos 2x}{(\pi - x)^2}$

6. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{4^{\sin x} - 4}{\ln \sin x}$

7. $\lim\limits_{x \to 0} \dfrac{x - \operatorname{Arctan} x}{x^2 \tan x}$

8. $\lim\limits_{x \to 2} \dfrac{1 - \sqrt{3 - x}}{4 - x^2}$

9. $\lim\limits_{x \to 0} \dfrac{x \ln(1 + x)}{1 - \cos x}$

10. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos x}{\sin 2x}$

11. $\lim\limits_{x \to \infty} \dfrac{\ln x^{10}}{x}$

12. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan x}{\tan 3x}$

13. $\lim_{x \to \infty} \frac{e^{2x}}{x^3}$

14. $\lim_{x \to \infty} \frac{x^2}{\ln x}$

15. $\lim_{x \to \infty} \frac{2x^2 + 1}{4x^2 + x}$

16. $\lim_{x \to \infty} \frac{\ln^2 x}{x}$

17 $\lim_{x \to \frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 3x}$

18. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

5.5 The Indeterminate Forms 0 $(\pm \infty)$ and $\infty - \infty$

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \pm\infty$, then the function defined by the product of $f(x)\, g(x)$ is said to have the indeterminate form $0\,(\pm\infty)$ at $x = a$. Forms of this type can be changed into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that LHR can be applied. To effect this change, we rewrite $f(x)\, g(x)$ into any of the following forms:

$$\frac{f(x)}{1/g(x)} \quad \text{or} \quad \frac{g(x)}{1/f(x)}$$

In general, one form is better than the other but the choice will depend upon the given product $f(x)\, g(x)$.

EXAMPLE 1. Evaluate $\lim_{x \to 0} x\cot x$

Solution: Let $f(x) = x$ and $g(x) = \cot x$. The limit of their product is of the type $0 \cdot \infty$ since $f(x) = 0$ and $g(x) = \infty$ at $x = 0$. Thus

$$\begin{aligned}
\lim_{x \to 0} x\cot x &= \lim_{x \to 0} \frac{x}{1/\cot x} \\
&= \lim_{x \to 0} \frac{x}{\tan x} && (= \tfrac{0}{0}) \\
&= \lim_{x \to 0} \frac{1}{\sec^2 x} && \text{(by LHR)} \\
&= \frac{1}{(1)^2} \\
&= 1
\end{aligned}$$

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the function defined by $f(x) - g(x)$ is said to have the indeterminate form $\infty - \infty$. This form can be changed to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebric manipulation so that LHR can be applied.

EXAMPLE 2. Evaluate $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

Solution: This limit is of the type $\infty - \infty$. since

$$\sec \frac{\pi}{2} = \infty \text{ and } \tan \frac{\pi}{2} = \infty.$$

$$\begin{aligned}
\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) &= \lim_{x \to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) && \text{(why?)} \\
&= \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} && (= \tfrac{0}{0}) \\
&= \lim_{x \to \frac{\pi}{2}} \frac{0 - \cos x}{- \sin x} && \text{(by LHR)}
\end{aligned}$$

$$= \lim_{x \to \frac{\pi}{2}} \cot x$$

$$= \cot \frac{\pi}{2}$$

$$= 0$$

EXERCISE 5.3

Evaluate each of the following limits:

1. $\lim_{x \to \infty} \left(e^{\frac{1}{x}} - 1\right) x$

2. $\lim_{x \to 0} \sin x \ln x$

3. $\lim_{x \to 0} \frac{\pi}{x} \tan \frac{\pi x}{2}$

4. $\lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x$

5. $\lim_{x \to \infty} x \sin \frac{4}{x}$

6. $\lim_{x \to 2} (4 - x^2) \tan x \frac{\pi x}{4}$

7. $\lim_{x \to 0} x \csc 2x$

8. $\lim_{x \to 0} (\csc x - \cot x)$

9. $\lim_{x \to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$

10. $\lim_{x \to 1} \left(\frac{2}{x^2 - 1} - \frac{1}{x-1}\right)$

$\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

$\lim_{x \to 0} \left(\frac{1}{1 - \cos x} - \frac{2}{\sin^2 x}\right)$

$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

$\lim_{x \to 0} \left(\frac{1}{\ln(1 + x)} - \frac{1}{\operatorname{Arctan} x}\right)$

...e Indeterminate Forms 0^0, 1^∞, and ∞^0

...e function defined by the expression $f(x)^{g(x)}$ may, at a ... value of x, assume any of the following indeterminate ...

0^0 if $f(x) = 0$ and $g(x) = 0$

1^∞ if $f(x) = 1$ and $g(x) = \infty$

∞^0 if $f(x) = \infty$ and $g(x) = 0$

...luate $\lim_{x \to a} f(x)^{g(x)}$ when any of the indeterminate forms above ...ned, we may perform the following steps;

Let $N = \lim_{x \to a} f(x)^{g(x)}$

Take the natural logarithm of both sides of the equation in (1)

$$\ln N = \ln \lim_{x \to a} f(x)^{g(x)}$$

$$= \lim_{x \to a} \ln f(x)^{g(x)} \qquad \text{by L12}$$

$$= \lim_{x \to a} g(x) \ln f(x) \qquad \text{by P3}$$

The limit at this point takes the form $0\cdot$ (we, therefore, apply the method used in Thus

3. $\ln N = \lim\limits_{x\to a} \dfrac{\ln f(x)}{1/g(x)}$. This limit is of the ty

4. Apply LHR to the right member of (3).

5. Suppose $\ln N = L$. Then $N = e^L$ where L is number. Therefore

$$\lim_{x\to a} f(x)^{g(x)} = e^L$$

EXAMPLE. Evaluate $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^x$

Solution; This limit is of the type 1^∞.

$$\text{Let } N = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$$

$$\text{Then } \ln N = \lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^x$$

$$= \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) \qquad (=\infty\cdot 0)$$

$$= \lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} \qquad \left(=\frac{0}{0}\right)$$

$$= \lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{2}{x}}\right)\left(\frac{-2}{x^2}\right)}{\frac{-1}{x^2}} \qquad \text{(by LHR)}$$

$$= \lim_{x\to\infty}\left(\frac{2}{1+\frac{2}{x}}\right)$$

$$= \frac{2}{1+0}$$

$$= 2$$

$$= e^2$$

nclude that

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^2$$

EXERCISE 5.4

e each of the following limits:

$\lim\limits_{x\to 0}\ (1+2x)^{\frac{3}{x}}$

$\lim\limits_{x\to 0}\ (4^x-1)^x$

$\lim\limits_{x\to 0}\ (\sin x)^x$

$\lim\limits_{x\to 0}\ (\csc x)^{\sin x}$

$\lim\limits_{x\to 1}\ x^{\frac{1}{1-x}}$

$(\tan x)^{\cos x}$

$\left(1+\dfrac{1}{x}\right)^{4x}$

8. $\lim_{x \to \infty} x^{e^{-x}}$

9. $\lim_{x \to 0} (x + \cos x)^{\frac{1}{x}}$

10. $\lim_{x \to 0} (1 + \sin x)^{\cot x}$

11. $\lim_{x \to \infty} \left(1 + 3x^{-1}\right)^{x}$

12. $\lim_{x \to 0} (1 + \tan x)^{\frac{1}{x}}$

13. $\lim_{x \to 0} \left(2 - e^{\frac{x}{2}}\right)^{\frac{4}{x}}$

14. $\lim_{x \to \frac{\pi}{4}} (\sin 2x)^{\tan 2x}$

15. $\lim_{x \to 0} (1 - x^2)^{\cot x}$

Chapter 6

The Differential

So far we have regarded the notation $\frac{dy}{dx}$ as a single symbol denote the limit of the quotient $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$ or to represent derivative of the function $y = f(x)$. Now we shall introduce extremely simple but useful concept called the *differential.* new concept will give meaning to the symbols dy and dx separately and in effect will permit us to consider the symbol the quotient of two differentials.

Differential: Definition and Interpretation

Consider a function defined by $y = f(x)$ where x is the independent variable. In Chapter 2, we introduced the symbol to denote the increment of x. Now we introduce the symbol which we call the *differential of x.* Similarly, we shall call the symbol *dy* as the *differential of y.* To give separate meanings to and dy, we shall adopt the following definitions of a function defined by the equation $y = f(x)$.

DEFINITION 6.1 $\quad dx = \Delta x$

In words, Definition 6.1 simply says that *the differential the independent variable is equal to the increment of the variable.*

DEFINITION 6.2 $\quad dy = f'(x)\, dx$

In words, Definition 6.2 states that *the differential of a function is equal its derivative multiplied by the differential of its independent variable.*

We emphasize that the differential dx is also an indep dent variable, i.e., it may be assigned any value whatsoev Therefore, from Definition 6.2, we see that the differential is a function of two independent variables, x and dx. It shou also be noted that while $dx = \Delta x$, $dy \neq \Delta y$ in general*.

Suppose $dx \neq 0$ and we divide both sides of the equati

$$dy = f'(x)\,dx$$

by dx. Then we get

$$\frac{dy}{dx} = f'(x)$$

Note that this time $\frac{dy}{dx}$ denotes the quotient of two differenti i.e., dy and dx. Thus the definition of the differential make possible to define the derivative of a function as the ratio of t differentials. That is,

$$f(x) = \frac{dy}{dx} = \frac{\text{differential of } y}{\text{differential of } x}$$

The differential may be given a geometric interpretati Consider again the equation $y = f(x)$ and let its graph be as sho in Fig. 6.1 Let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be two points the curve. Draw the tangent to the curve at P. Through Q, dr a perpendicular to the x-axis and intersecting the tangent at Then draw a line through P, parallel to the x-axis and intersect the perpendicular through Q at R. Let θ be the inclination of t tangent PT.

*Recall that $\Delta y = f(x + \Delta x) - f(x)$.

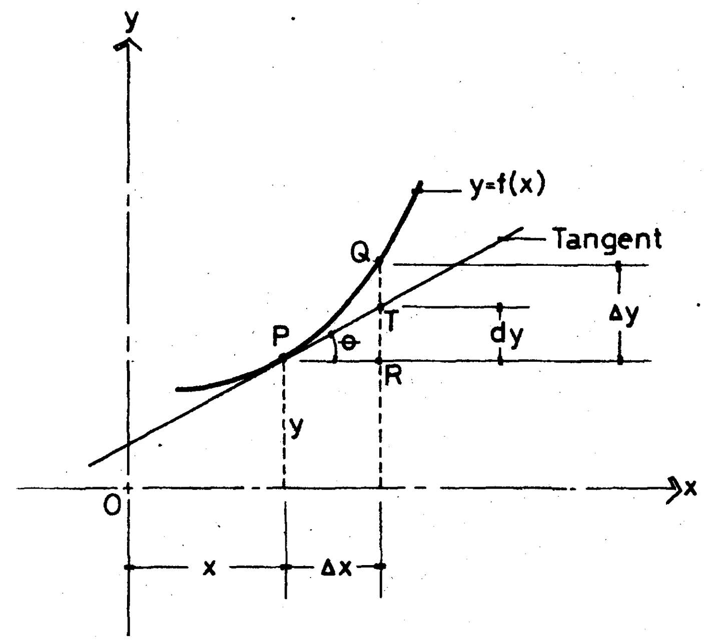


FIG. 6.1

From analytic geometry, we know that

$$\text{slope of } PT = \tan\theta \tag{1}$$

But in triangle PRT, we see that

$$\tan\theta = \frac{RT}{PR} = \frac{RT}{\Delta x} \tag{2}$$

However, $\Delta x = dx$ by Definition 6.1. Hence (2) becomes

$$\tan\theta = \frac{RT}{dx} \tag{3}$$

From Chapter 2, recall that the value of the derivative of $y = f(x)$ at P is equal to the slope of the tangent at that same point P Hence in Fig. 6.1,

$$\text{slope of } PT = f'(x)$$

Substituting (3) and (4) in (1), we get

$$f'(x) = \frac{RT}{dx}$$

Solving for RT in (5), we obtain

$$RT = f'(x)dx$$

But by Definition 6.2, the right member of (6) is dy. He

$$RT = dy$$

We see that dy is the increment of the ordinate of the tang
line corresponding to an increment of Δx in x whereas Δy is
corresponding increment of the curve for the same increm
in x. We also note that the derivative $\frac{dy}{dx}$ or $f'(x)$ gives the slo
of the tangent while the differential dy gives the *rise of the t*
gent line.

6.2 Differential Formulas

Since we have already considered $\frac{dy}{dx}$ as the ratio of two
ferentials, then the differentiation formulas in Chapter 2, (Sec. 2
may now be expressed in terms of differentials by multiply
both sides of the equations by dx. Thus

d1. $d(c) = 0$

d2. $d(x) = dx$

d3. $d(cu) = cdu$

d4. $d(u + v) = du + dv$

d5. $d(uv) = udv + vdu$

d6. $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$

$$d(u^n) = nu^{n-1}\,du$$

$$d(\sqrt{u}) = \frac{du}{2\sqrt{u}}$$

$$d\left(\frac{1}{u^n}\right) = \frac{-n}{u^{n+1}}\,du$$

ymbol *d* is regarded as the *operator* which indicates
of finding the differential of a function.

ifferentiation formulas for other types of functions
in Chapter 4 may also be expressed in terms of diffe-

1 Find dy if (1) $y = x^3 - 4x^2 + 5x$ and (2) $y = \frac{2x}{3x-1}$

tion: (1) $y = x^3 - 4x^2 + 5x$

$$dy = d(x^3 - 4x^2 + 5x)$$

$$= 3x^2\,dx - 8xdx + 5dx$$

$$= (3x^2 - 8x + 5)dx$$

(2) $y = \frac{2x}{3x-1}$

$$dy = d\left(\frac{2x}{3x-1}\right)$$

$$= \frac{(3x-1)(2dx) - (2x)(3dx)}{(3x-1)^2}$$

$$= \frac{(6x-2)dx - 6xdx}{(3x-1)^2}$$

$$= \frac{(6x-2-6x)dx}{(3x-1)^2}$$

$$= \frac{-2dx}{(3x-1)^2}$$

Note: In practice, we simply get
of the right member of the
multiply it by dx. Thus for
lem, the solution will simp
as follows:

$$y = x^3 - 4x^2 + 5x$$

$$dy = (3x^2 - 8x + 5)dx$$

EXAMPLE 2. Find $\frac{dy}{dx}$ by means of differentials if xy

Solution: $xy + \sin x = \ln y$

$$xdy + ydx + \cos x dx = \frac{1}{y}dy$$

$$xydy + y^2 dx + y\cos x dx = dy$$

$$xy\frac{dy}{dx} + y^2 + y\cos x = \frac{dy}{dx}$$

$$xy\frac{dy}{dx} - \frac{dy}{dx} = -y^2 - y\cos x$$

$$(xy-1)\frac{dy}{dx} = -y(y+\cos x)$$

$$\frac{dy}{dx} = \frac{-y(y+\cos x)}{xy-1}$$

EXERCISE 6.1

implify whenever possible.

$y = 3x^4 - 4x^2 + 2x$

$y = \sqrt[3]{12x}$

$y = 4e^{2x}$

$y = \frac{3x}{x-2}$

$y = x\ln x$

$y = x^3 e^{-2x}$

$y = \frac{1}{2}x + \frac{1}{4}\sin 2x$

$y = e^x \cos^2 x$

$y = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32}$

$y = \text{Arctan}(\tan 3x)$

y use of differentials:

$xy + \text{Arctan}(xy) = 0$

$2\ln(x^2 + y^2) = \text{Arctan}\frac{y}{x}$

$x^3 + xy^2 = y$

$e^y = \sin(x-y)$

$4^x + 4^y = 32$

$\ln x + 2\ln y = xy$

6.3 Applications of the differential

Let us consider again the graph of $y = f(x)$ in Fig. 6, that the difference $\Delta y - dy$ is represented by the direct ment TQ. The magnitude of this difference can made as please by making Δx sufficiently small. In other words, Δx, we expect dy and Δy to be nearly equal. We are effect, that dy may be used to approximate the value of while the true or exact value of y at $x + \Delta x$ is

$$f(x + \Delta x) = y + \Delta y,$$

its approximate value for small Δx is

$$f(x + \Delta x) \doteq y + dy$$

where (recall chapter 5) the symbol "$\doteq$" is read "is appr equal to".

EXAMPLE 1. Compute $\sqrt{37}$ approximately by use of tials.

Solution: Let $\sqrt{37} \doteq y + dy$

and $37 = x + dx$

where x is a perfect square nearest to 37

Obviously $37 = 36 + 1$

Hence $x = 36$ and $dx = 1$

Let $y = \sqrt{x}$

Then $dy = \dfrac{dx}{2\sqrt{x}}$

For $x = 36$, $y = \sqrt{36} = 6$ and sinc

we have $dy = \dfrac{1}{2\sqrt{36}} = \dfrac{1}{12}$

Therefore, $\sqrt{37} \doteq 6 + \dfrac{1}{12} = \dfrac{73}{12}$

Note that if we have been asked to find $\sqrt{32}$ or $\sqrt{40}$, then our approximation by use of differentials would not have been so good. Why?

EXAMPLE 2. If $y = x^3 + 2x^2 - 3$, find the approximate value of y when $x = 2.01$.

Solution: The exact value is $y + \Delta y$ but since we are simply asked to find the approximate value, then we shall solve for $y + dy$. Note that if we write $2.01 = 2 + 0.01$, then we are considering 2.01 as the result of applying an increment of $\Delta x = dx = 0.01$ to an original value of $x = 2$.

Since $y = x^3 + 2x^2 - 3$ (1)

then $dy = (3x^2 + 4x)dx$ (2)

When $x = 2$, then from (1)

$$y = 8 + 8 - 3 = 13$$

and when $x = 2$ and $dx = 0.01$, then from (2),

$$dy = (12 + 8)(0.01) = 0.20$$

Therefore, the required approximation is

$$y + dy \doteq 13 + 0.20 = 13.20$$

EXAMPLE 3. Each side of a square is increased by Δx. Find the approximate and true increase of the area A of the square.

Solution: The approximate increase in A is dA and the true increase is ΔA. The area of the original square (ABCD in Fig. 6.2) is

$$A = x^2$$

By differentials,

$$dA = 2x\,\Delta x = 2xdx$$

Thus 2xdx gives the approximate increase in A. In Fig. 6.2, this approximate increase is given by

$$dA = \text{area of DCEF} + \text{area of ADHI.}$$

The true increase in A is

$$\Delta A = (x + \Delta x)^2 - x^2$$

$$= x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2$$

$$= 2x\,\Delta x + (\Delta x)^2$$

In Fig. 6.2, this true increase in A is ΔA = ar DCEF + area of ADHI + area of DFGH.

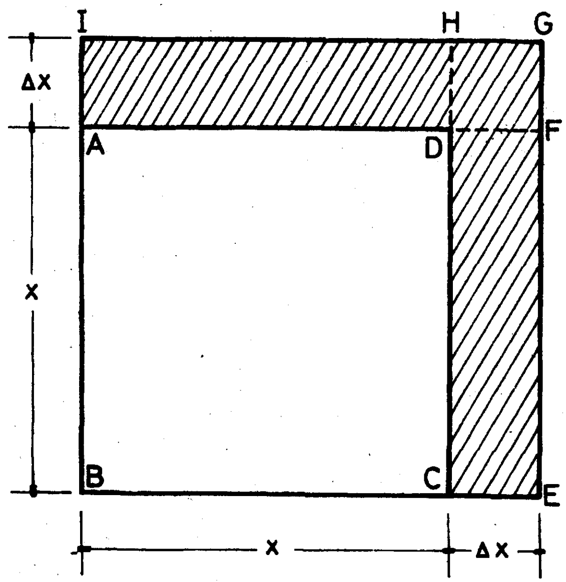


FIG. 6.2

The differentials are also used in approximate computation of certain quantities due to small errors in measurement. For instance, if $y = f(x)$, then an error dx in the measurement of x leads to an approximate error dy in the quantity y. The approximate relative error (RE) in y is the ratio $\frac{dy}{y}$ and the approximate percentage error (PE) in y is $\frac{dy}{y}$ (100).

EXAMPLE 4. The radius of a circle is measured to be 10 cm with an error of 0.05 cm. Find the relative error in the computed area.

Solution: We are asked to find the RE in the area A when $r = 10$ cm and $dr = 0.05$ cm. Hence

$$A = \pi r^2 = \pi (10)^2 = 100\pi \text{ cm}^2$$

$$dA = 2\pi\, rdr = 2\pi\,(10)\,(0.05) = \pi \text{ cm}^2$$

Therefore,

$$RE = \frac{dA}{A}$$

$$= \frac{\pi}{100\pi}$$

$$= 0.01$$

EXAMPLE 5. Find the approximate percentage error in the computed volume V of a cube of edge x cm if an error of 2% is made in measuring an edge.

Solution: We are asked to find the PE in V when the PE in x is $\frac{dx}{x}(100) = 2\%$.

Since $\qquad V = x^3$

then $\qquad dV = 3x^2\,dx$

and the percentage error in the volume V is

$$PE = \frac{dV}{V}(100)$$

$$= \frac{3x^2\,dx}{x^3}(100)$$

$$= 3\left(\frac{dx}{x}\cdot 100\right)$$

$$= 3\,(2\%)$$

$$= 6\%$$

EXERCISE 6.2

Find the approximate value of the following by use of differentials

1. $\sqrt{626}$
2. $\sqrt[3]{215}$
3. $(82)^{3/4}$
4. $(63.4)^{2/3}$
5. $\sqrt[5]{31.6}$
6. $(1.98)^4$
7. $\ln(2.3)$ if $\ln 2 = 0.6931$
8. $e^{2.4}$ if $e^2 = 7.3891$

Use differentials to find the approximate value of y:

9. $y = (2x-1)^4$ when $x = 0.98$
10. $y = x^4 - 2x^3 + 3x^2 + x - 1$ when $x = 1.02$

the following problems by use of differentials:

The circumference of a circle is 100 cm. If the radius is increased by 0.1 cm, find the approximate increase in the area.

If an error of 1.5% is made in measuring the side of an equilateral triangle, find the percentage error made in the computed area.

The radius of a sphere is measured to be 4 cm with an error of 0.002 cm. Find the relative error in the computed volume.

In a right circular cone, the radius of the base is half as long as the altitude. If an error of 2% is made in measuring the radius, find the percentage error made in the computed volume.

Find the approximate surface area of a sphere of radius 4.02 in.

Find the approximate area of a square when the side is 4.01 cm.

A circular hole 4 inches in diameter and 1 foot deep in a block of iron is drilled out to increase its diameter to 4.1 in. Find the approximate volume of the metal removed.

The diameter of a circle is to be measured and its area computed. If the diameter can be measured with a maximum error of 0.002 cm. and the allowable error in the area is 0.01 cm^2, find the diameter of the largest circle for which the specifications are met.

For a right circular cylinder of height 25 cm., the radius was measured as 20 cm with an error of 0.05 cm. Find the approximate percentage error in the computed volume.

PTER

Derivatives from Parametric Equations, Radius and Center of Curvature

analytic geometry, we have learned that a curve may also ibed analytically by a pair of equations of the form

$$x = g(t) \ , \ y = h(t).$$

quations are called *parametric equations* of the curve and variable *t* is called a *parameter*. For example, the circle $- a^2$ may be represented by the parametric equations

$$x = a\cos t \ , \ y = a\sin t$$

the parameter t is the angle between the x-axis and the the point (x, y).

atives in Parametric Form

y f(x) be a function whose parametric representation in the form

$$x = g(t) \ , \ y = h(t).$$

in Chapter 3 (see sec. 3.7) that

$$\frac{dx}{dt} = \text{rate of change of x with respect to t}$$

$$\frac{dy}{dt} = \text{rate of change of y with respect to t}$$

ently the rate of change of y with respect to x of a func- ed by $x = g(t)$, $y = h(t)$ will be given by

$$\text{E(7.1)} \qquad \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}$$

Next, we consider the problem of finding the seco tive of a function defined by the parametric equations Chapter 2 (sec. 2.7), the second derivative is defined as

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

and by the Chain Rule, we may write the equation ab form

$$E(7.2) \qquad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{d}$$

EXAMPLE 1. If $x = t^3 - 1$, $y = t^2 + t$, find $\frac{dy}{dx}$ and

Solution: Since $x = t^3 - 1$, $y = t^2 +$

then $\frac{dx}{dt} = 3t^2$, $\frac{dy}{dt} = 2t + 1$

By E(7.1),

$$\frac{dy}{dx} = \frac{2t+1}{3t^2}$$

and by E(7.2),

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{2t+1}{3t^2}\right)\frac{dt}{dx}$$

$$= \frac{(3t^2)(2) - (2t+1)(6t)}{9t^4} \cdot \frac{1}{3t}$$

$$= \frac{-2(t+1)}{9t^5}$$

EXAMPLE 2. If $x = 2\sin\theta$, $y = 1 - 4\cos\theta$, find $\frac{dy}{dx}$

Solution: $x = 2\sin\theta$, $y = 1 - 4\cos\theta$

$$\frac{dx}{d\theta} = 2\cos\theta \qquad \frac{dy}{d\theta} = 4\sin\theta$$

Then $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{4\sin\theta}{2\cos\theta}$$

$$= 2\tan\theta$$

and $$\frac{d^2y}{dx^2} = \frac{d}{d\theta}(2\tan\theta)\frac{d\theta}{dx}$$

$$= 2\sec^2\theta\,\frac{1}{2\cos\theta}$$

$$= \sec^3\theta$$

EXERCISE 7.1

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and simplify whenever possible.

1. $x = t^3 + 1$, $y = t^2 + 1$
2. $x = t^{-3}$, $y = t^3 + 3t$
3. $x = u^3 + 1$, $y = 4u^2 - 4u$
4. $x = \sqrt{u+2}$, $y = u^2 - 3$
5. $x = 1 + \cos t$, $y = \sin^2 t$
6. $x = 1 - \ln t$, $y = t - \ln t$
7. $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$
8. $x = \cos^3\theta$ $y = \sin^3\theta$
9. $x = e^{\phi}$, $y = 2e^{-\phi}$

10. $x = \phi e^{\phi}$, $y = e^{\phi}$

11. Find the slope of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ when $\theta = \frac{\pi}{2}$.

12. Find the slope of the curve $x = e^{\theta}\sin\theta$, $y = e^{\theta}\cos\theta$ when $\theta = \frac{\pi}{3}$.

13. Find the equation of the tangent to the curve $x = 2\sin t$, $y = \cos 2t$ when $t = \frac{\pi}{6}$

14 Find the equation of the tangent to the curve $x = \ln t$, $y = t^{-1}$ when $t = 2$.

7.2 Differential of Arc Length

Let a curve be defined parametrically by the equations

$$x = g(t) \ , \quad y = h(t)$$

where g and h are differentiable functions of t. Also let the arc

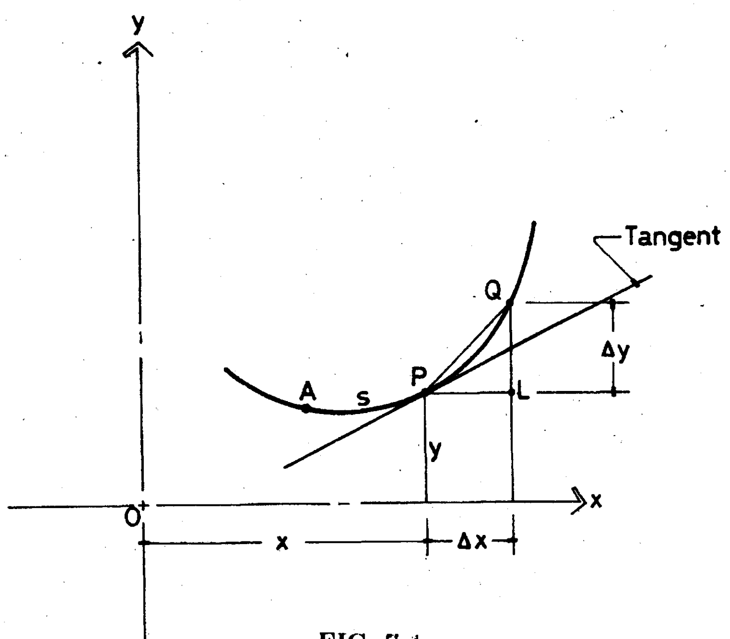


FIG. 7.1

length from a fixed point A to a variable point $P(x, y)$ be denoted by the small letter s (Fig. 7.1). Consider a nearby point $Q(x + \Delta x, y + \Delta y)$ and let Δs be the arc PQ. Since s is a function of t, then we may wish to find the rate at which s changes with respect to t, i.e., $\frac{ds}{dt}$.

By definition,
$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \qquad (1)$$

But we may express $\frac{\Delta s}{\Delta t}$ as

$$\frac{\Delta s}{\Delta t} = \frac{\Delta s}{PQ} \cdot \frac{PQ}{\Delta t} \qquad (2)$$

where the chord PQ in Fig. 7.1 is the hypotenuse of the right triangle PLQ. We note that

$$PQ = \sqrt{(\Delta x)^2 + (\Delta y)^2} \qquad (3)$$

Then (2) may be written in the form

$$\frac{\Delta s}{\Delta t} = \frac{\Delta s}{PQ} \cdot \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t}$$

$$\text{or} \quad \frac{\Delta s}{\Delta t} = \frac{\Delta s}{PQ} \cdot \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \qquad (4)$$

Substituting (4) in (1), we get

$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{PQ} \cdot \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \qquad (5)$$

$$= \lim_{\Delta t \to 0} \frac{\Delta s}{PQ} \lim \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

It can be shown that the limit of $\frac{\Delta s}{PQ}$ as $\Delta t \to 0$ is unity, that is, $\lim_{\Delta t \to 0} \frac{\Delta s}{PQ} = 1$. Hence (5) becomes

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad (6)$$

Multiplying both sides of (6) by dt, we obtain

$$E(7.3) \qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

where ds denotes the differential of arc length. From E(7.3), we can obtain the following forms for ds:

$$E(7.4) \qquad ds = \sqrt{(dx)^2 + (dy)^2}$$

$$E(7.5) \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx$$

$$E(7.6) \qquad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy$$

If the equation of the curve is given in the polar form $r = f(\theta)$ then the differential of arc length is given by

$$E(7.7) \qquad ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \; d\theta$$

E(7.7) can easily be obtained by use of the familiar relations between rectangular and polar coordinates, that is,

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

7.3 Radius of Curvature

We have seen in our previous discussion, that the concept of the derivative is related to the tangent to a curve. Another concept of geometric interest is that of *curvature*. Consider the curve $y = f(x)$ as shown in Fig. 7.2. Let s be the length of the arc of the curve between a fixed point A and a variable point P. Denote the slope-angle* of the tangent T to the curve at P by ϕ.

*The angle between the tangent and the x-axix.

...te of change of ϕ with respect to s is called the *curvature* ... curve at P and is denoted by K. That is

$$K = \frac{d\phi}{ds}$$

...lue of K is either positive or negative. If $K > 0$, the curve ...ave upward at P. On the other hand. if $K < 0$, the curve is ...ve downward at P. However, it is customary to consider K ...itive. For this reason, we rewrite our defining equation ...as

$$E(7.8) \qquad K = \left|\frac{d\phi}{ds}\right|$$

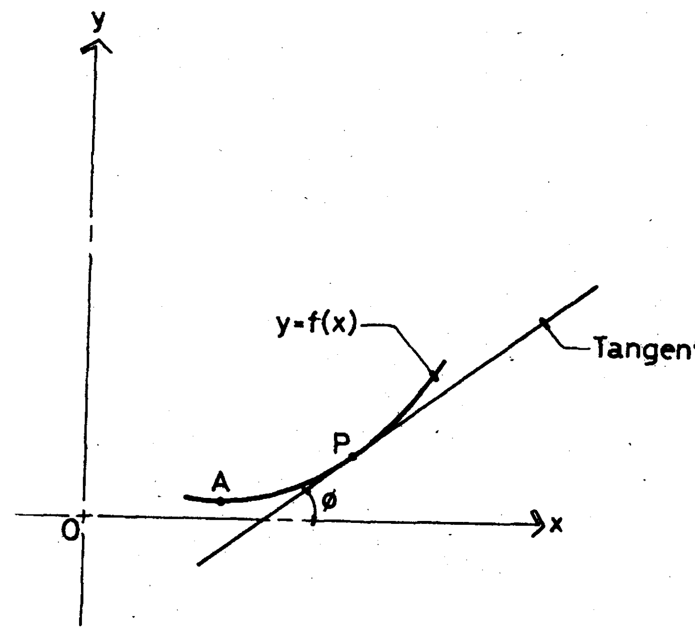


FIG. 7.2

...The reciprocal of the curvature is called the *radius of cur-*... and is denoted by R. That is

$$R = \frac{1}{K}$$

Substituting (3) and (5) in (2) and simplifying, we get

$$\frac{ds}{d\phi} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Substituting (6) in E(7.9), we obtain

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

The absolute value symbol is not needed in the numer $\left(\frac{dy}{dx}\right)^2 > 0$. To simplify notation, we may write (7) in

$$\text{E(7.10)} \qquad R = \frac{[1 + (y')^2]^{3/2}}{|y''|}$$

where $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$.

If the equation of the curve is given by $x = g(y)$, th fining equation for R takes the form

$$\text{E(7.11)} \qquad R = \frac{[1 + (x')^2]^{3/2}}{|x''|}$$

where $x' = \frac{dx}{dy}$ and $x'' = \frac{d^2x}{dy^2}$.

When the equation of a curve is given parametricall form

$$x = g(t) \ , \ y = h(t)$$

the expression for K in E(7.8), we have the defining formula as

$$\text{E(7.9)} \qquad R = \left|\frac{ds}{d\phi}\right|$$

We next derive a formula for R which is applicable if the ion of the curve is given in the form $y = f(x)$. By the chain

$$\frac{ds}{d\phi} = \frac{ds}{dx} \cdot \frac{dx}{d\phi} \qquad (1)$$

quation (1) may be written in the form

$$\frac{ds}{d\phi} = \frac{ds/dx}{d\phi/dx} \qquad (2)$$

E(7.5), we obtain

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad (3)$$

definition of slope,

$$\tan\phi = \frac{dy}{dx}$$

g for ϕ ,

$$\phi = \text{Arctan}\,\frac{dy}{dx} \qquad (4)$$

entiating (4) with respect to x,

$$\frac{d\phi}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} \qquad (5)$$

the radius of curvature can be shown to be

E(7.12) $R = \frac{\left[(g')^2 + (h')^2\right.}{|g'h'' - g'}$

where $g' = \frac{dx}{dt}$, $g'' = \frac{d^2x}{dt^2}$, $h' = \frac{dy}{dt}$, and h''

It can also be shown that the radius of curva
curve $r = f(\theta)$ is given by

E(7.13) $R = \frac{\left[r^2 + (r')^2\right.}{|r^2 + 2(r')^2}$

where $r' = \frac{dr}{d\theta}$ and $r'' = \frac{d^2r}{d\theta^2}$. The proof of E(7.13) is
cise. However, we shall state and prove a theorem
giving the proof of E(7.13).

THEOREM: If α is the angle between the
and the tangent to the curve
point $P(r, \theta)$, then

E(7.14) $\tan\alpha = \frac{r}{r'}$

where $r' = \frac{dr}{d\theta}$

We shall now give the proof of the theorem
be the angle between the x -axis and the radius
ϕ the inclination of the tangent to the curve at P
between the radius vector and the tangent (see F
θ, ϕ, and α are related by the equation

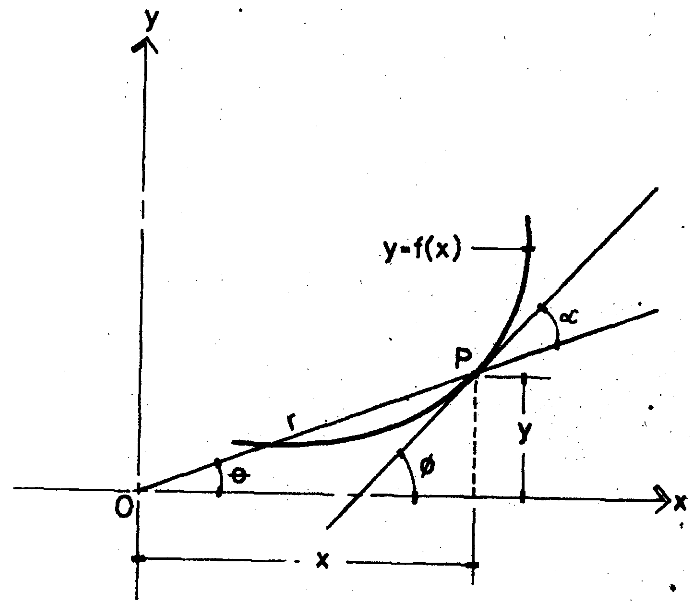


FIG. 7.3

$$\alpha = \phi - \theta \quad (1)$$

follows from (1) that

$$\tan\alpha = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta} \quad (2)$$

from Fig. 7.3 and by the definition of slope, we have

$$\tan\theta = \frac{y}{x} \quad (3)$$

$$\tan \quad \frac{dy}{dx} \quad (4)$$

stituting (3) and (4) in (2) and simplifying

$$\tan\alpha = \frac{xdy - ydx}{xdx + ydy} \quad (5)$$

From analytic geometry, we know that

$$x = r\cos\theta \ , \ y = r\sin\theta \quad (6)$$

By differentiation, we obtain from (6) the following equations

$$dx = -r\sin\theta d\theta + \cos\theta dr \ , \ dy = r\cos\theta + \sin\theta dr \quad (7)$$

Substituting (6) and (7) in (5), we obtain

$$\tan\alpha = \frac{rd\theta}{dr}$$

or $$\tan\alpha = \frac{r}{\dfrac{dr}{d\theta}}$$

Letting $r' = \dfrac{dr}{d\theta}$, we obtain E(7.14).

EXAMPLE 1. Find the radius of curvature of $y = x^3$ at $x = 1$

Solution: Since $y = x^3$

then $y' = 3x^2$ and $y'' = 6x$

when $x = 1$, $y' = 3$ and $y'' = 6$

Then by E(7.10),

$$R = \frac{[1 + (3)^2]^{3/2}}{|6|}$$

$$= \frac{5\sqrt{10}}{3}$$

EXAMPLE 2. Find the radius of curvature of the curve $x = \sin t - 1$, $y = 2\cos t + 3$ at $t = \frac{\pi}{2}$.

Solution: $x = \sin t - 1$ $\quad y = 2\cos t + 3$

$x' = \cos t \qquad y' = -2\sin t$

$x'' = -\sin t \qquad y'' = -2\cos t$

When $t = \frac{\pi}{2}$,

$x' = 0 \qquad y' = -2$

$x'' = -1 \qquad y'' = 0$

Substituting these values in E(7.12), we get $R = 4$.

EXERCISE 7.2

Find the radius of curvature at the given point:

1. $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ at $(0, a)$
2. $y = x^4$ at $(1, 1)$
3. $y = \sin x$ at $(\frac{\pi}{2}, 1)$
4. $x = \dfrac{8}{y^2 + 4}$ at $(2, 0)$
5. $x = e^y - 2y$ at $(1, 0)$
6. $y^2 = 4x$ at $(1, 2)$
7. $16x^2 + 25y^2 = 400$ at one end of the minor axis
8. $x = e^t$, $y = e^{-t}$ at $t = 0$
9. $x = t^2 - 2t$, $y = 1 - 4t$ at $t = 1$

10. $x = 2\sin t$, $y = \cos 2t$ at $t = \frac{\pi}{6}$

11. $x = e^t \sin t$, $y = e^t \cos t$ at $t = 0$

12. $r = a\cos 3\theta$ at $\theta = \frac{\pi}{6}$

13. $r = a(1 - \cos\theta)$ at $\theta = \pi$

Find the radius of curvature at any point on the curve.

14. $y = \ln\sin x$

15. $x^{2/3} + y^{2/3} = a^{2/3}$

16. $y^2 = 8x$

17. $b^2x^2 + a^2y^2 = a^2b^2$

18. $r^2 = a^2\cos 2\theta$

19. $x = \cos\phi + \phi\sin\phi$, $y = \sin\phi - \phi\cos\phi$

20. $x^2 + y^2 = a^2$

21. Derive E(7.12)

22. Derive E(7.13)

7.4 Center of Curvature

Through any point P(x, y) on the curve y = f(x), we ca construct a tangent circle whose radius r is equal to the radi of curvature R of the curve at P as shown in Fig. 7.4. This uniq circle is called the *circle of curvature* and its center is called th *center of curvature* of the curve. This center of curvature whi has coordinates (h, k) lies on the normal to the curve at P. We sh show how the coordinates (h, k) can be expressed in terms of th coordinates (x, y) of P.

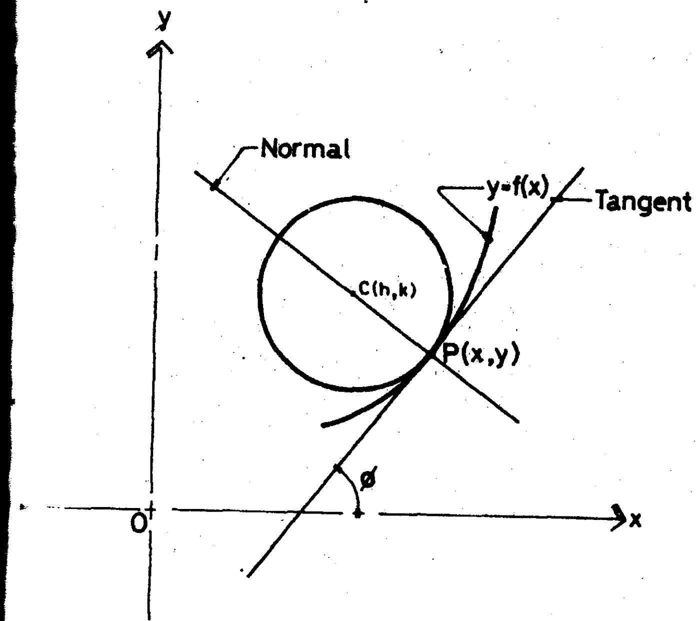


FIG. 7.4

equation of the circle of curvature is

$$(x - h)^2 + (y - k)^2 = R^2 \qquad (1)$$

ting E(7.10) in (1), we have

$$(x - h)^2 + (y - k)^2 = \frac{[1 + (y')^2]^3}{(y'')^2} \qquad (2)$$

e slope of the tangent at P is y', then the slope of the nor- $1/y'$. The equation of the normal is

$$y - k = -\frac{1}{y'}(x - h) \qquad (3)$$

(2) and (3) simultaneously for h and k, we obtain

$$\text{E(7.15)} \qquad h = x - \frac{y'[1 + (y')^2]}{y''}$$

E(7.16) $k = y + \dfrac{1 + (y')^2}{y''}$

EXAMPLE. Find the center of curvature of the cu
at the point (1,0).

Solution: Since $y = e^{-x}$

then $y' = -e^{-x}$

$y'' = e^{-x}$

At (1,0) $y' = -1$

$y'' = 1$

By E(7.15), $h = 2$

By E(7.16), $k = 3$

Hence the center of curvature is (2, 3)

EXERCISE 7.3

Find the center of curvature at the point indicated.

1. $y = \frac{1}{2}(e^x + e^{-x})$ at (0, 1)
2. $y = \sin x$ at $(\frac{\pi}{2}, 1)$
3. $y^2 = 8x$ at $(\frac{1}{2}, 2)$
4. $xy = 4$ at (2, 2)
5. $y = \ln x$ at
6. $y = \frac{1}{2}\tan 2x$

Partial Differentiation

preceding chapters have been concerned with the dif-
ions of functions with one independent variable. In this
we shall study differentiation of functions of several
. Examples of these functions are the following familiar

$$V = \pi r^2 h \tag{1}$$

$$A = \frac{1}{2} ab\sin\phi \tag{2}$$

(1) expresses the volume V of a right circular cylinder
of the base radius r and the altitude or height h. That
function of two variabbles, r and h. Formula (2) shows
area A of an oblique triangle is a function of three varia-
ly a, b, and ϕ.

l Derivative

$z = f(x, y)$ be a function of two independent variables
. If y is held constant, then z becomes temporarily a func-
the single variable x. From this point of view, we can
the derivative of z with respect to x by employing
for ordinary differentiation of functions with single
The derivative found in this manner is called the *partial*
of z with respect to x and the process involved is called
fferentiation. The derivative of z with respect to x is de-
any of the following symbols:

$$\frac{\partial f}{\partial x}, \; z_x \;, \; f_x(x, y) \;, \; f_x$$

, if x is held constant, then z becomes temporarily a
of y. As a result, we can compute the partial deriva-
with respect to y and this derivative may be denoted by
e following symbols:

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, z_y, f_y(x, y), f_y$$

It should be noted that the symbol $\frac{\partial z}{\partial x}$ (or $\frac{\partial z}{\partial y}$) c thought of as a fraction since neither of the symbols (or ∂z and ∂y) has a separate meaning*. The symbol means to differentiate partially with respect to x whatev it. The symbol $\frac{\partial}{\partial y}$ is interpreted in like manner.

Formally, the definition of partial derivatives can as follows:

If $z = f(x, y)$, then the partial derivative of z with x is symbolically defined as

$$E(8.1) \quad \frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

and the partial derivative of z with respect to y is defined

$$E(8.2) \quad \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

This definition can be extended to functions of two variables. In general, with function f of severa x, y, z, ... , there is a partial derivative with resp independent variable, i.e., f_x, f_y, f_z, . . .

EXAMPLE 1 f $z = x^2y + 4x + 3y$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by u E(8.1) and E(8.2) respectively.

Solution: Let $z = f(x, y) = x^2y + 4x + 3y$

Then $f(x + \Delta x, y) = (x + \Delta x)^2\, y + 4(x +$

*The symbol ∂ which is a special form of the Greek letter delta w by Jacobi (1804-1851).

and $f(x, y + \Delta y) = x^2(y + \Delta y) + 4x + 3(y + \Delta y)$

By E(8.1).

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2\, y + 4(x + \Delta x) + 3y] - [x^2y + 4x + 3y]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2xy\,\Delta x + (\Delta x)^2 y + 4\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2xy + \Delta x . y + 4)$$

$$= 2xy + 4$$

By E(8.2)

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{[x^2(y + \Delta y) + 4x + 3(y + \Delta y)] - [x^2y + 4x + 3y]}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2\Delta y + 3\Delta y}{\Delta y}$$

$$= \lim_{\Delta y \to 0} (x^2 + 3)$$

$$= x^2 + 3$$

In practice, we compute $\frac{\partial z}{\partial x}$ in the example above by considering y as constant and then differentiate with respect to x by applying the rules for ordinary differentiation.

Thus

Since $z = x^2y + 4x + 3y$

Then $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2y + 4x + 3y)$

$$= 2xy + 4$$

To find $\frac{\partial z}{\partial y}$, we treat x as constant and differentiate with respect to y. Thus

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 y + 4x + 3y)$$

$$= x^2 + 3$$

EXAMPLE 2. If $z = x\sin y + y\sin x$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: Considering y as constant and differentiating with respect to x.

$$\frac{\partial z}{\partial x} = \sin y + y\cos x$$

Considering x as constant and differentiating with respect to y,

$$\frac{\partial z}{\partial y} = x\cos y + \sin x$$

EXAMPLE 3. If $u = x^2 + yz^2 + xz$, find u_x, u_y, and u_z.

Solution: Considering y and z as constants and differentiating with respect to x,

$$u_x = 2x + z$$

Considering x and z as constants and differentiating with respect to y,

$$u_y = z^2$$

Considering x and y as constants and differentiating with respect to z,

$$u_z = 2yz + x$$

It should be noted that before performing any partial differentiation of functions of several variable, it is important to know first which of the variables are considered or held constants.

8.2 Geometric Interpretation of Partial Derivative

We shall now give a simple geometric interpretation to the concept of partial derivative. Let the graph of a surface* defined by the equation $z = f(x, y)$ be as shown in Fig. 8.1. Let $P(x_o, y_o, z_o)$ be a point on the surface. Then the plane passing through P and parallel to the xz plane has the equation $y = y_o$. The intersection of the surface $z = f(x, y)$ and the plane $y = y_o$ is the curve APB as shown in Fig. 8.1. As a point moves along the curve APB, its coordinates x and z vary while y remains constant. The slope of the tangent line at P represents the rate at which z changes with respect to x. Hence the partial derivative $\frac{\partial z}{\partial x}$ is the slope of the tangent to the curve of intersection APB at the point P.

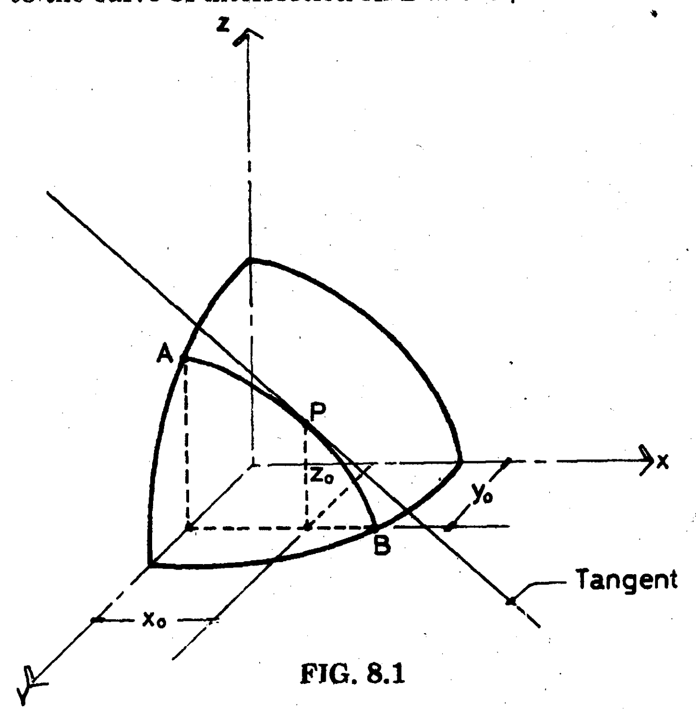


FIG. 8.1

*The student who is not familiar with graphs of surfaces should refer to any text on analytic geometry for a brief review.

The equations of the tangent at P are

E(8.3) $z - z_o = m_o(x - x_o)$, y =

where m_o = value of $\frac{\partial z}{\partial x}$ at P.

In Fig. 8.2, the curve CPD is the curve of intersection of surface $z = f(x, y)$ and the plane $x = x_o$. As a point moves al the curve CPD, y and z vary while x remains constant. He $\frac{\partial z}{\partial y}$ is the slope of the tangent to the curve CPD at the point

The equations of the tangent at P are

E(8.4) $z - z_o = m_o(y - y_o)$, x =

where m_o = value of $\frac{\partial z}{\partial y}$ at P.

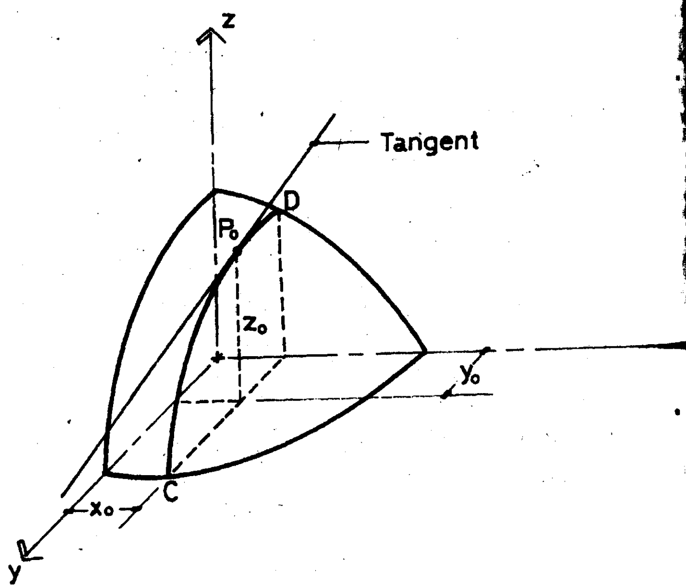


FIG. 8.2

I.E. Find the equations of the tangent to the parabola $z = x^2 + 3y^2$, $y = 1$ at the point (2, 1, 7).

ution: The parabola which is the curve of intersection of the surface $z = x^2 + 3y^2$ and the plane $y = 1$ is represented by the curve APB in Fig. 8.3 Its vertex is at point A. Since y is constant, we differentiate partially the equation $z = x^2 + 3y^2$ with respect of x. Thus

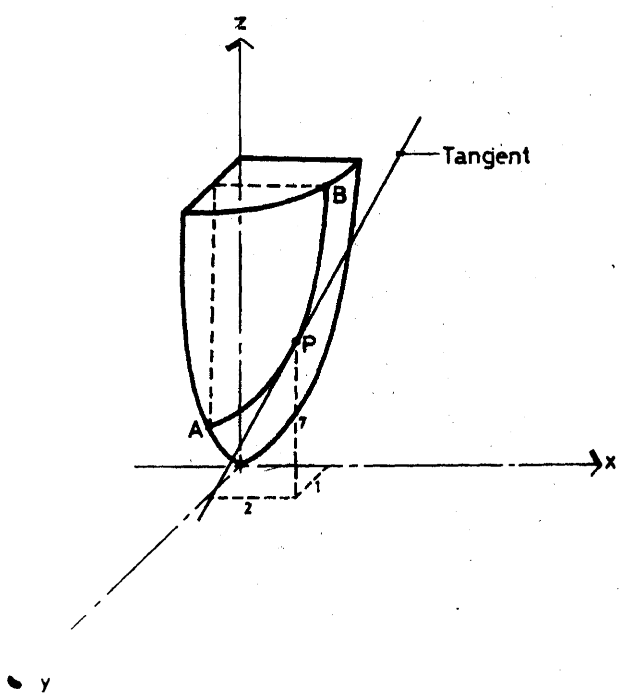
$$\frac{\partial z}{\partial x} = 2x$$

At (2, 1, 7), we have $\frac{\partial z}{\partial x} = 2(2) = 4$. Hence $m_o = 4$, $x_o = 2$, and $z_o = 7$. Then by E(8.3), the equations of the tangent at P are

$$z - 7 = 4(x - 2),\ y = 1$$

or

$$z = 4x - 1\ ,\ y = 1$$



(1) Formula 12 can be extended to the sum of number of differentials.

(2) Formula I3 tells us that a constant may be mo the integral sign. (Note: You can not do this ble.)

(3) Formula I4 is used for finding the integral o of a function. Note that it holds for any re n except $n = -1$. Note further that if $u = x$, fies to

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

EXAMPLE 1. Evaluate $\int (5x^4 + 3x^2 + 6)dx$

Solution:

$$\int (5x^4 + 3x^2 + 6)dx = \int 5x^4\,dx + \int 3x^2\,dx + \int 6dx$$

$$= 5\int x^4\,dx + 3\int x^2\,dx + 6\int$$

$$= \frac{5x^5}{5} + c_1 + \frac{3x^3}{3} + c_2 + 6x +$$

$$= x^5 + x^3 + 6x + (c_1 + c_2 +$$

$$= x^5 + x^3 + 6x + C$$

where* $C = c_1 + c_2 + c_3$. In practice, integral is simply evaluated in the manner:

$$\int (5x^4 + 3x^2 + 6)dx = \frac{5x^5}{5} + \frac{3x^3}{3} + 6$$

$$= x^5 + x^3 + 6x$$

*Constants can always be combined into a single constant.

LE 2. $\int (3x + 4)^2 dx = \int (9x^2 + 24x + 16)dx$

$$= \frac{9x^3}{3} + \frac{24x^2}{2} + 16x + C$$

$$= 3x^3 + 12x^2 + 16x + C$$

LE 3. $\int \left(\frac{4}{x^3} + \frac{2}{x}\right)dx = \int \left(4x^{-3} + \frac{2}{x}\right)dx$

$$= \int 4x^{-3}dx + 2\int \frac{dx}{x}$$

$$= \frac{4x^{-2}}{-2} + 2\ln|x| + C$$

$$= -\frac{2}{x^2} + 2\ln|x| + C$$

EXERCISE 9.1

the following:

$x^2 - 4x + 5)dx$

1) $(3x + 4)dx$

$\sqrt{x} - 1)dx$

$\frac{+ 4)dx}{\sqrt{x}}$

$\frac{+ 4x - 3\, dx}{x^2}$

$\sqrt{x} - 2x\sqrt{x})dx$

$\frac{8}{2}$ dx

8. $\int \frac{(1+\sqrt[3]{x})^2}{\sqrt[3]{x}}\,dx$

9. $\int \sqrt{x^4 - 2x^3 + x^2}\,dx$

10. $\int \left(\frac{5}{\sqrt{x}} - \frac{3}{x^2} + \frac{2}{x^4}\right) dx$

9.3 Integration by Substitution

Some integrals can not be evaluated readily cation of the standard integration formulas. T evaluating such integrals leans heavily on what *method of substitution.** This method involves ble, say from x to another variable u. The purpo a new variable is to bring the problem to a form dard formula can be applied. This integration b justified by the so called *Chain Rule for integratio* briefly state below.

Let F(u) be a function whose derivative $\frac{d}{du}F(u) = f(u)$. If u is a differentiable function

then

E(9.4) $\qquad \int f(u)du = \int f[h(x)]\, h'(x)$

Let us now prove E(9.4). We are given that

$$\frac{dF(u)}{du} = f(u)$$

Then $\qquad dF(u) = f(u)du$

Integrating both sides of (2)

*Other methods of integration by substitution will be discu

$$\int dF(u) = \int f(u)du \qquad (3)$$

Also, since $\qquad u = h(x) \qquad (4)$

Then $\qquad \frac{du}{dx} = h'(x) \qquad (5)$

By the Chain Rule for differentiation,

$$\frac{dF(u)}{dx} = \frac{dF(u)}{du}\cdot\frac{du}{dx} \qquad (6)$$

By (1), (4) and (5), equation (6) becomes

$$\frac{dF(u)}{dx} = f[h(x)]\,h'(x)$$

or $\qquad dF(u) = f[h(x)]\,h'(x)dx \qquad (7)$

Integrating both sides of (7)

$$\int dF(u) = \int f[h(x)]\,h'(x)dx \qquad (8)$$

Comparing (3) and (8), we see that

$$\int f(u)du = \int f[h(x)]\,h'(x)dx$$

which is what we wanted to prove.

AMPLE 1. Evaluate $\int (3x+4)^2\,dx$

Solution: Let $u = 3x + 4$. Then $du = 3dx$ or $\frac{1}{3}du = dx$. Then the given integral becomes

$$\int (3x+4)^2\,dx = \int u^2 \cdot \frac{1}{3}du$$

$$= \frac{1}{3}\int u^2\,du \qquad \text{by I3}$$

Solution: Perhaps to a beginner, this problem may quite difficult. It seems not easy to re which expression should be equated to time, we need that little "trick" we m above. To do the trick, all we need are algebraic manipulations. Thus

$$\int (4x^3 + x)\sqrt{4x^2 + 1}\, dx = \int x(4x^2 + 1)\,(4x^2 +$$

$$= \int (4x^2 + 1)^{\frac{3}{2}}\, xdx \qquad \text{(let } u =$$

$$= \frac{1}{8} \cdot \frac{(4x^2 + 1)^{\frac{5}{2}}}{\frac{5}{2}} + C \qquad \text{(nf} =$$

$$= \frac{(4x^2 + 1)^{\frac{5}{2}}}{20} + C$$

Consider this time the problem of evaluating the of the rational fraction

$$\frac{f(x)}{g(x)}$$

where the degree of $f(x) \geq$ degree of $g(x)$. To evaluate t must first carry out the indicated division until the remaind lower degree than the denominator. That is,

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}$$

where $Q(x)$ = quotient

$R(x)$ = remainder of lower degree than $g(x)$

Therefore, we have

$$\int \frac{f(x)}{g(x)}\, dx = \int \left[Q(x) + \frac{R(x)}{g(x)} \right] dx$$

$$= \int Q(x)dx + \int \frac{R(x)}{g(x)}\, dx$$

Evaluate $\displaystyle\int \frac{2x^2 - 6x + 4}{x - 3}\, dx$

Here $f(x) = 2x^2 - 6x + 4$ and $g(x) = x - 3$. Carrying out the indicated division, we get

$$\frac{2x^2 - 6x + 4}{x - 3} = 2x + \frac{4}{x - 3}$$

where $Q(x) = 2x$ and $R(x) = 4$

Therefore,

$$\int \frac{2x^2 - 6x + 4}{x - 3}\, dx = \int 2xdx + \int \frac{4}{x - 3}\, dx$$

$$= x^2 + 4\ln|x - 3| + C$$

EXERCISE 9.2

of the following indefinite integrals.

dx

$)^{3}(6x^2 + 4)dx$

$1)^4 dx$

dx / 2x + 1

dx

dx

7. $\int \frac{x^2\,dx}{(x^3-1)^4}$

8. $\int \frac{\sqrt{\ln 4x}\,dx}{x}$

9. $\int \frac{dx}{x\ln^2 x}$

10. $\int \frac{e^{2x}\,dx}{\sqrt{1+e^{2x}}}$

11. $\int \frac{dx}{e^x - 1}$

12. $\int \sin^3 x\cos x dx$

13. $\int \cos^4 x\sin x dx$

14. $\int \frac{\sin 2x dx}{(1-\cos 2x)^4}$

15. $\int \sqrt{1+2\sin 3x}.\cos 3x dx$

16. $\int \frac{\cos 4x dx}{\sin^3 4x}$

17. $\int \frac{\sec^2 x dx}{a + b\tan x}$

18. $\int \cot^3 4x\csc^2 4x dx$

19. $\int \sqrt{\tan 3x}\,\sec^2 3x dx$

20. $\int \frac{1+e^x}{1-e^x}\,dx$

21. $\int \frac{3x^2+14x+13}{x+4}\,dx$

$\frac{x+5}{x-1}\,dx$

$\frac{2x^3-2x}{x^2+1}\,dx$

$\frac{x^3+4x}{x^2-1}\,dx$

on of Trigonometric Functions

ndard formulas for evaluating the integrals of the six functions are given below. The first two can be differentiation and the remaining four may be proved

$$\int \sin u du = -\cos u + C$$

$$\int \cos u du = \sin u + C$$

$$\int \tan u du = -\ln|\cos u| + C$$

$$\int \cot u du = \ln|\sin u| + C$$

$$\int \sec u du = \ln|\sec u + \tan u| + C$$

$$\int \csc u du = -\ln|\csc u + \cot u| + C$$

Since $d(-\cos u + C) = \sin u du$, then we have verified that T1 is correct.

$$\int \tan u du = \int \frac{\sin u}{\cos u}\,du$$

EXERCISE 9.3

Evaluate each of the following:

1. $\int \sec 5x \tan 5x dx$
2. $\int \frac{dx}{\sin x \cos x}$
3. $\int \frac{\sin x + \cos x}{\sin^2 x} dx$
4. $\int \sec^2 (4x - 3) dx$
5. $\int \frac{dx}{\sin \frac{1}{2}x \cot \frac{1}{2}x}$
6. $\int \frac{dx}{1 - \cos x}$
7. $\int \frac{\cos^3 x}{1 - \sin x} dx$
8. $\int \frac{\cos 4x}{\sin 2x} dx$
9. $\int (1 + \tan x)^2 dx$
10. $\int x^2 \cos 4x^3 dx$
11. $\int \frac{\cos 6x dx}{\cos^2 3x}$
12. $\int \sin 2x \sec x dx$
13. $\int \frac{\sin 2x dx}{2 \sin x \cos^2 x}$
14. $\int (\cot x + \tan x)^2 dx$

15. $\int \frac{4 \sin^2 x \cos^2 x}{\sin 2x \cos 2x} dx$
16. $\int \frac{dx}{\tan 5x}$
17. $\int \frac{dx}{\sin 3x \tan 3x}$

9.5 Integration of Exponential Functions

The following formulas for evaluating the integrals of exponential functions can be proved by differentiation.

E1. $\int e^u du = e^u + C$

E2. $\int a^u du = \frac{a^u}{\ln a} + C$, $\quad a > 0, a \neq 1$

EXAMPLE 1. Evaluate $\int e^{4x} dx$

Solution: Here $u = 4x$ and $du = 4dx$. Hence $nf = \frac{1}{4}$. By E1,

$$\int e^{4x} dx = \frac{1}{4} e^{4x} + C$$

EXAMPLE 2. Evaluate $\int 4^{3x} dx$

Solution: This takes the form a^u with $a = 4$ and $u = 3x$. Hence $du = 3dx$ and $nf = \frac{1}{3}$. By E2,

$$\int 4^{3x} dx = \frac{1}{3} \cdot \frac{4^{3x}}{\ln 4} + C$$

$$= \frac{4^{x3}}{3 \ln 4} + C$$

$$= \frac{4^{3x}}{\ln 64} + C$$

EXERCISE 9.4

Evaluate the following:

1. $\int \frac{dx}{e^{2x}}$
2. $\int (e^{3x} + 1)^2 dx$
3. $\int e^{\sin 4x} \cos 4x\, dx$
4. $\int \frac{e^x - 4e^{-x}}{e^x}\, dx$
5. $\int \sqrt{e^{3x}}\, dx$
6. $\int 2^{4x}\, dx$
7. $\int 5^{3-2x}\, dx$
8. $\int \sqrt[3]{4^{2x}}\, dx$
9. $\int 3^x 2^x dx$
10. $\int \frac{3^x}{2^x}\, dx$

9.6 Integration of Hyperbolic Functions

The following formulas are used for evaluating the inte of hyperbolic functions. Formulas H1 to H6 may be veri by differentiation. For example, H1 is correct since d(sinhu + C coshudu. The student may give the proof of H7 and H8.

H1. $\int \cosh u\, du = \sinh u + C$

H2. $\int \sinh u\, du = \cosh u + C$

$\int \operatorname{sech}^2 u\, du = \tanh u + C$

$\int \operatorname{csch}^2 u\, du = -\coth u + C$

$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$

$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$

$\int \tanh u\, du = \ln|\cosh u| + C$

$\int \coth u\, du = \ln|\sinh u| + C$

E 1. $\int \cosh(4x+3)dx = \frac{1}{4}\sinh(4x+3) + C$ by H1

E 2. $\int x \tanh x^2\, dx = \frac{1}{2}\ln\left|\cosh x^2\right| + C$ by H7

EXERCISE 9.5

the following:

nh $(3x - 1)dx$

$(\cosh 4x + \sinh 2x)dx$

$h^2(1 - x^2)\,.\,x dx$

$h\ \frac{1}{4}x \tanh \frac{1}{4}x dx$

$\frac{h^2(\ln x)dx}{x}$

$th(1 - 2x)dx$

$h\ \frac{1}{2}x \coth \frac{1}{2}x dx$

$\frac{h4x}{h2x}\ dx$

(or two sines) can be reduced to an integral of
rence) of two cosines. That is

P1. $\int 2\sin u\cos v\,dx = \int [\,\sin(u+v)$

P2. $\int 2\cos u\cos v\,dx = \int [\,\cos(u+v)$

P3. $\int 2\sin u\sin v\,dx = \int [\,\cos(u-v)$

The right member of P1 is then evaluated
and those of P2 and P3 by T2. Consider the fol

EXAMPLE 1. Evaluate $\int \cos 6x\cos 2x\,dx$

Solution: We have the product of two co
and v = 2x. Hence, we shall use P

$$\int \cos 6x\cos 2x\,dx = \frac{1}{2}\int 2\cos 6x\cos 2x\,dx$$
$$= \frac{1}{2}\int (\cos 8x + \cos 4x)\,dx$$
$$= \frac{1}{2}\left(\frac{1}{8}\sin 8x + \frac{1}{4}\sin 4x\right)$$
$$= \frac{1}{16}\sin 8x + \frac{1}{8}\sin 4x + C$$

EXAMPLE 2. Evaluate $\int 3\sin 5x\cos 4x\,dx$

Solution: We have the product of a sine
u = 5x and v = 4x. Hence, we
by using P1.

$$\int 3\sin 5x\cos 4x\,dx = \frac{3}{2}\int 2\sin 5x\cos 4x\,dx$$
$$= \frac{3}{2}\int (\sin 9x + \sin x)\,dx$$
$$= \frac{3}{2}\left(-\frac{1}{9}\cos 9x - \cos x\right) +$$
$$= -\frac{1}{6}\cos 9x - \frac{3}{2}\cos x + C$$

EXERCISE 10.1

of the following:

nx dx

7x dx

3)cos(x + 5)dx

cos $\frac{1}{4}$ xdx

$2\pi)\cos(x+\pi)dx$

2x)sin(2x + 3)dx

3x dx

$\frac{\pi}{2})\cos(x+\frac{\pi}{4})dx$

$\frac{\pi}{3})\sin(2x-\frac{\pi}{6})dx$

$\frac{\pi}{4})\cos(x+\frac{\pi}{2})dx$;

$$= \frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

$$= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

EXERCISE 10.2

Evaluate each of the following:

1. $\int \sin^5 x\cos^4 x\,dx$
2. $\int \sin^3 2x\sqrt{\cos 2x}\,dx$
3. $\int \sin^4 3x\cos^3 3x\,dx$
4. $\int \frac{\cos^5 x}{\sin^3 x}\,dx$
5. $\int \sin^4 x\cos^2 x\,dx$
6. $\int \sin^6 x\cos^4 x\,dx$
7. $\int (\sqrt{\sin x} + \cos x)^2\,dx$
8. $\int (1 + \cos 4x)^2\,dx$
9. $\int (\sin 3x + \cos 2x)^2\,dx$
10. $\int \sin^2 7x\,dx$
11. $\int \cos^2 4x\,dx$
12. $\int \cos^7 x\,dx$
13. $\int \sin^3 2x\,dx$
14. $\int \sin^3 x\cos^5 x\,dx$
15. $\int \sin^7 x\cos^3 x\,dx$

ers of Tangents and Secants

nsider the trigonometric integral of the form

$$\int \tan^m v\sec^n v\,dx$$

n 1, we evaluate the integral by T9 (see Section 9.4).
ny number and n = 2, we evaluate the integral by the or-
thod of substitution used in Section 9.3. For example,
u = tan x, we can show that

$$\ldots x\sec^2 x\,dx = \frac{\tan^4 x}{4} + C$$

s section, we shall consider the following cases:

1: *When m is any number and n is a positive even integer greater than 2*, we may write

$$\tan^m v\sec^n v = (\tan^m v\sec^{n-2} v)\sec^2 v$$

and then use the identity

$$\sec^2 v = 1 + \tan^2 v$$

to reduce the given integral to the form

$$\int (\text{sum of powers of } \tan v)\sec^2 v\,dx$$

which is now integrable by I4 and with u = tanv.

1. Evaluate $\int \tan^3 x\sec^4 x\,dx$

ion:

$$\begin{aligned} \ldots\sec^4 x\,dx &= \int \tan^3 x\sec^2 x\sec^2 x\,dx \\ &= \int \tan^3 x(1 + \tan^2 x)\sec^2 x\,dx \\ &= \int (\tan^3 x + \tan^5 x)\sec^2 x\,dx \\ &= \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C \end{aligned}$$

EXERCISE 10.3

Evaluate the following:

1. $\int \tan^2 2x\sec^4 2x\,dx$;
2. $\int \tan^5 x\sec^4 x\,dx$
3. $\int \sqrt{\tan x}\,\sec^6 x\,dx$
4. $\int \tan^7 x\,dx$
5. $\int \qquad ^{\frac{1}{2}} x\,dx$
6. $\int \sec^6 4x\,dx$
7. $\int (\sec x + \tan x)^2\,dx$
8. $\int (1 - \sec 4x)^2\,dx$
9. $\int \left(\frac{\sec 3x}{\tan 3x}\right)^4 dx$
10. $\int \tan^3 x\sec^{3/2} x\,dx$
11. $\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx$
12. $\int \sqrt{\sec 5x}\,\sec^2 5x\tan 5x\,dx$

10.4 Powers of Cotangents and Cosecants

The technique involved in evaluating the integral

$$\int \cot^m v\csc^n v\,dx$$

where v is a differentiable function of x, is similar to that for evaluating the integral

$$\int \tan^m v\sec^n v\,dx.$$

The identity $\csc^2 v = 1 + \cot^2 v$ or $\cot^2 v = \csc^2 v - 1$ is used to reduce the original expression into an integrable form. It also consists of three possible cases and it is left to the student to write down the procedure for evaluating each case.

EXERCISE 10.4

Evaluate each of the following:

1. $\int \cot^4 x\csc^4 x\,dx$
2. $\int \cot^3 x\csc^3 x\,dx$
3. $\int \cot^5 4x\,dx$
4. $\int (\csc^2 x - 1)^2\,dx$
5. $\int \sqrt{\cot 3x}\,\csc^4 3x\,dx$
6. $\int \frac{dx}{\sin^6 4x}$
7. $\int \frac{\cos^5 2x\,dx}{\sin^8 2x}$
8. $\int (\csc 4x + \cot 4x)^2\,dx$
9. $\int \frac{\csc^4 x}{\cot^6 x}\,dx$
10. $\int \frac{dx}{\tan^4 6x}$