LESSON 1
COMPLEX NUMBERS

This topic is a preparation to more complex engineering mathematics such as Laplace Transforms, Fourier Series and Fourier Transform.
At the end of the discussion , the students are expected to
a. apply the properties of complex numbers in getting roots,
b. solve problems involving powers and logartithm
c. solve problems involvoving trigonometric and hyperbolic functions.
d. Apply Complex Numbers in Engineering Mechanics such as component and resultant of forces .

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1.1 Definition

The Complex Number z is the number $z = x + jy$
where:
x is the real part
jy is the imaginary part
and $j = \sqrt{-1}$
(Note j and i are used interchangeably)

1.2 Integral Powers of j

$j = \sqrt{-1}$
$j^2 = -1$
$j^3 = j^2 j = -1j = -j$
$j^4 = (j^2)^2 = (-1)^2 = 1$

Example 1
Simplify the following:
a. j^{243}
b. j^{132}
c. j^{182}
Solution:
a. Divide 243 by 4 and get the remainder.
$243/4 = 60\ ¾$
$j^{243} = (j^4)^{60} j^3 = 1(-j) = -j$
b. $132/4 = 33 \quad j^{132} = (j^4)^{33} = 1$
c. $182/4 = 45\ 2/4$, then $j^{182} = j^2 = -1$

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1.3 Arithmetic of Complex Numbers

Example 2

Let A = 4 + 3j B = -8 – 7j

Addition:

A + B = 4 + 3j + (-8 – 7j)

= -4 – 4j

Subtraction:

A – B = 4 + 3j – (-8 – 7j)

= 12 + 10j

Mutiplication:

AB = (4 + 3j)(-8 – 7j)

= 4(-8) + 4(-7j) + 3j(-8) – $21j^2$

= -32 – 28j – 24j + 21

= -11 – 52j

Division:

$$A/B = \frac{4+3j}{-8-7j}$$

Multiply numerator and denominator by conjugate of - 8 – 7j i.e. – 8 + 7j

$$A/B = \frac{4+3j}{-8-7j} \cdot \frac{-8+7j}{-8+7j}$$

$$A/B = \frac{(4+3j)(-8+7j)}{(-8-7j)(-8+7j)} = \frac{-32-24j+28j+21j^2}{64-49j^2}$$

= (-53 + 4j)/ 113

= -53/113 + 4j/113

1.4 Theorems on Complex Numbers

a. If a + bj = 0 then a= 0 and b = 0.

b. If a + bj = c + dj then a = b and c = d.

c. If (a + bj)(c + dj) = 0 then
a + bj = 0 or c + dj = 0 or both are 0 .

Example 3

Find the values of x and y if

(3x + 4y) + (7x – 2y + 5)j = 4 + 2j

Solution:

3x + 4y = 4 and 7x – 2y + 5 = 2 or 7x – 2y = -3

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Solve simultaneously:

$3x + 4y = 4$

$7x - 2y = -3$

$x = -2/17$ and $y = 37/34$

Example 4:

Find the two square roots of $-3 - 4j$.

Solution:

Let $x + jy = \sqrt{(-3 - 4j)}$

Then $(x + jy)^2 = -3 - 4j$

$x^2 + 2xyj + j^2y^2 = -3 - 4j$

$x^2 - y^2 + 2xy\,j = -3 - 4j$

Using Theorem b of 1.4

$x^2 - y^2 = -3 \qquad (1)$

$2xy = -4 \qquad (2)$

From (2) $y = -2/x$

Plug this to (1)

$x^2 - (-2/x)^2 = -3$

$x^4 - 4 = -3x^2$

$x^4 + 3x^2 - 4 = 0$

$(x^2 - 1)(x^2 + 4) = 0$

x is real thus $x^2 - 1 = 0$

$(x - 1)(x + 1) = 0$

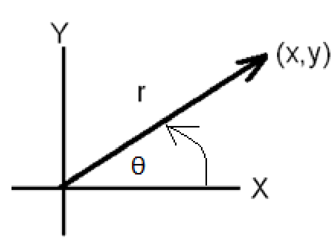
$x = 1, -1 \quad y = -2, 2$

The roots are $(1, -2)$, $(-1, 2)$

The 2 square roots are $1 - 2i$ and $-1 + 2i$

1.5 Geometric Representation of Complex Numbers.

A complex number $x + jy$ is geometrically represented by a directed line segment (vector)



that joins the origin and (x,y). (Figure 1)

Figure 1

where $x = r\cos\theta \qquad y = r\sin\theta$

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tan θ = y/x

$r = \sqrt{x^2 + y^2}$

θ is the argument

r is the modulus or absolute value of z (|z|)

θ is usually given in principal value.

$-180\ deg < \theta < 180$ deg

1.6 Polar Form of Complex Number

z = x + jy

z = r cos θ + j r sin θ

z = r (cos θ + j sin θ)

or z = r cis θ or z = r∟θ

1.7 Principal Value of θ

The argument is in its principal value if

$-180^0 \le \theta \le 180^0$

1.8 Exponential Form:

From Eulers Formula:

$e^{j\theta} = \cos\theta + j\sin\theta$

Then: $r(\cos\theta + j\sin\theta) = re^{j\theta}$

where θ is expressed in radians.

Exponential Form of z = x + jy

is $z = re^{j\theta}$ where θ is in radians

and $r = \sqrt{x^2 + y^2}$

The General Polar form of z = x + jy

is z = r ∟ (θ + k(360)) where

k = 0, ±1, ±2 ….

The General Exponential Form is

$z = re^{j(\theta \pm k(2\pi))}$ where

k = 0, ±1, ±2 …. and θ is in radians

(Note: + means counterclockwise rotation and – means clockwise rotation)

Example 4

Find the polar . general polar , exponential and general exponential form of - 5 – 7j .

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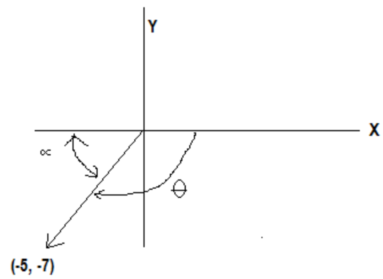

Solution: Let $z = -5 - 7j$

$x = -5$, $y = -7$ (3rd quadrant)

$r^2 = x^2 + y^2 = (-5)^2 + (-7)^2$

$r = 8.6023$

$\tan \alpha = y/x = 7/5$ (Assume x and y are both +)

$\propto = 54.46^0$

Then $\theta = -180 + 54.46^0 = -125.54^0$

(quadrant 3) Note θ is the principal value.

Polar Form:

$\mathbf{z = 8.6023 \angle -125.54^0}$

General Polar Form:

$\mathbf{z = 8.6023 \angle [-125.54 + k(360)]}$

Exponential Form:

Convert -125.54^0 to rad

$-125.54^0 = -2.191$ rad

Exponential Form:

$\mathbf{z = 8.6023\, e^{-2.191j}}$

General Exponential Form

$\mathbf{z = 8.6023\, e^{-2.191j + k(2\pi)}}$

1.10 Phasor Algebra

Let $A = r_1 \angle \theta$ and $B = r_2 \angle \alpha$

It can be shown that:

$AB = (r_1 \angle \theta)(r_2 \angle \alpha)$

$= r_1 r_2 \angle (\theta + \alpha)$

$A/B = r_1/r_2 \angle (\theta - \alpha)$

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Example 5

Let A = $20\angle 30^0$ and B = $14\angle -15^0$

C = $13\angle 124^0$

Evaluate (Express the result in rectangular form)

a. ABC

b. B/C

c. A/(B + C)

Solution:

a. ABC = ($20\angle 30^0$)($14\angle -15^0$)($13\angle 124^0$)

= 3640 ∟ (30 – 15 + 124)

= 3640 ∟ (139)

= 3640 (cos 139 + j sin 139)

= -2747.14 + 2388.05 j

b. B/C = $14\angle -15^0$ / $13\angle 124^0$

= 14/13 ∟ (-15 – 124)

= 14/13 ∟ -139^0

= 14/13 (cos -139 + j sin – 139)

= -0.813 – 0.707

c. A/ (B + C)

B + C = $14\angle -15^0$ + $13\angle 124^0$ =

14 (cos (-15) + j sin (-15)) + 13 (sin 124 + j sin 124)

= 13.52 – 3.62j + - 7.27 + 10.78 j

= 6.25 + 7.16i = $9.504\angle 48.88^0$

Then A/ (B + C) =

$20\angle 30^0$ / $9.504\angle 48.88^0$ =

2.104∟ (30 – 48.88) = 2.104∟ -18.88

= 1.99- 0.68j

Example 6

Evaluate $\frac{e^{2j} + e^{j}}{2j}$

Solution:

$e^{2j} + e^{j}$ = 1∟(2 x 57.3^0) + 1 ∟ (1 x 57.3^0)

Note: 1 rad = 57.3 deg

= 1 ∟114.6^0 + 1 ∟ 57.3^0

= -0.416 + 0.909 j + 0.54 + 0.841j

= 0.124 + 1.75j

Then $\frac{e^{2j} + e^{j}}{2j}$ = (0.124 + 1.75j)/ (2j)

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$= 0.875 - 0.062j$

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